JOHN BIRD

HIGHER ENGINEERING MATHEMATICS

SIXTH EDITION

Newnes

Higher Engineering Mathematics

In memory of Elizabeth

Higher Engineering Mathematics

Sixth Edition

John Bird, BSc (Hons), CMath, CEng, CSci, FIMA, FIET, MIEE, FIIE, FCollT

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Preface

This **sixth edition of *'Higher Engineering Mathematics'*** covers essential mathematical material suitable for students studying **Degrees, Foundation Degrees, Higher National Certificate and Diploma courses in Engineering disciplines**.

In this edition the material has been ordered into the following **twelve convenient categories**: number and algebra, geometry and trigonometry, graphs, complex numbers, matrices and determinants, vector geometry, differential calculus, integral calculus, differential equations, statistics and probability, Laplace transforms and Fourier series. **New material** has been added on logarithms and exponential functions, binary, octal and hexadecimal, vectors and methods of adding alternating waveforms. Another feature is that a **free Internet download** is available of a sample (over 1100) of the further problems contained in the book.

The primary aim of the material in this text is to provide the fundamental analytical and underpinning knowledge and techniques needed to successfully complete scientific and engineering principles modules of Degree, Foundation Degree and Higher National Engineering programmes. The material has been designed to enable students to use techniques learned for the analysis, modelling and solution of realistic engineering problems at Degree and Higher National level. It also aims to provide some of the more advanced knowledge required for those wishing to pursue careers in mechanical engineering, aeronautical engineering, electronics, communications engineering, systems engineering and all variants of control engineering.

In ***Higher Engineering Mathematics 6th Edition***, theory is introduced in each chapter by a full outline of essential definitions, formulae, laws, procedures etc. The theory is kept to a minimum, for **problem solving** is extensively used to establish and exemplify the theory. It is intended that readers will gain real understanding through seeing problems solved and then through solving similar problems themselves.

Access to software packages such as Maple, Mathematica and Derive, or a graphics calculator, will enhance understanding of some of the topics in this text.

Each topic considered in the text is presented in a way that assumes in the reader only knowledge attained in BTEC National Certificate/Diploma, or similar, in an Engineering discipline.

'Higher Engineering Mathematics 6th Edition' provides a follow-up to *'Engineering Mathematics 6th Edition'*.

This textbook contains some **900 worked problems**, followed by over **1760 further problems (with answers)**, arranged within **238 Exercises**. Some **432 line diagrams** further enhance understanding.

A **sample of worked solutions** to over 1100 of the further problems has been prepared and can be **accessed free via the Internet** (see next page).

At the end of the text, a list of **Essential Formulae** is included for convenience of reference.

At intervals throughout the text are some **19 Revision Tests** (plus two more in the website chapters) to check understanding. For example, Revision Test 1 covers the material in Chapters 1 to 4, Revision Test 2 covers the material in Chapters 5 to 7, Revision Test 3 covers the material in Chapters 8 to 10, and so on. An **Instructor's Manual**, containing full solutions to the Revision Tests, is available free to lecturers adopting this text (see next page).

Due to restriction of extent, five chapters that appeared in the fifth edition have been removed from the text and placed on the website. For chapters on Inequalities, Boolean algebra and logic circuits, Sampling and estimation theories, Significance testing and Chi-square and distribution-free tests (see next page).

'Learning by example' is at the heart of *'Higher Engineering Mathematics 6th Edition'*.

JOHN BIRD
Royal Naval School of Marine Engineering,
HMS Sultan,
formerly University of Portsmouth
and Highbury College, Portsmouth

Free web downloads

Extra material available on the Internet at: www.booksite.elsevier.com/newnes/bird.

It is recognised that the **level of understanding of algebra** on entry to higher courses is often inadequate. Since algebra provides the basis of so much of higher engineering studies, it is a situation that often needs urgent attention. Lack of space has prevented the inclusion of more basic algebra topics in this textbook; it is for this reason that some algebra topics – solution of simple, simultaneous and quadratic equations and transposition of formulae – have been made available to all via the Internet. Also included is a Remedial Algebra Revision Test to test understanding. To access the Algebra material visit the website.

Five extra chapters

Chapters on Inequalities, Boolean Algebra and logic circuits, Sampling and Estimation theories, Significance testing, and Chi-square and distribution-free tests are available to download at the website.

Sample of worked Solutions to Exercises

Within the text (plus the website chapters) are some 1900 further problems arranged within 260 Exercises. A sample of over 1100 worked solutions has been prepared and can be accessed free via the Internet. To access these worked solutions visit the website.

Instructor's manual

This provides fully worked solutions and mark scheme for all the Revision Tests in this book (plus 2 from the website chapters), together with solutions to the Remedial Algebra Revision Test mentioned above. The material is available to lecturers only. To obtain a password please visit the website with the following details: course title, number of students, your job title and work postal address.

To download the Instructor's Manual visit the website and enter the book title in the search box.

Syllabus Guidance

This textbook is written for **undergraduate engineering degree and foundation degree courses**; however, it is also most appropriate for **HNC/D studies** and three syllabuses are covered.
The appropriate chapters for these three syllabuses are shown in the table below.

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Engineering Mathematics
1.	Algebra	×		
2.	Partial fractions	×		
3.	Logarithms	×		
4.	Exponential functions	×		
5.	Hyperbolic functions	×		
6.	Arithmetic and geometric progressions	×		
7.	The binomial series	×		
8.	Maclaurin's series	×		
9.	Solving equations by iterative methods		×	
10.	Binary, octal and hexadecimal		×	
11.	Introduction to trigonometry	×		
12.	Cartesian and polar co-ordinates	×		
13.	The circle and its properties	×		
14.	Trigonometric waveforms	×		
15.	Trigonometric identities and equations	×		
16.	The relationship between trigonometric and hyperbolic functions	×		
17.	Compound angles	×		
18.	Functions and their curves		×	
19.	Irregular areas, volumes and mean values of waveforms		×	
20.	Complex numbers		×	
21.	De Moivre's theorem		×	
22.	The theory of matrices and determinants		×	
23.	The solution of simultaneous equations by matrices and determinants		×	
24.	Vectors		×	
25.	Methods of adding alternating waveforms		×	

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Engineering Mathematics
26.	Scalar and vector products		×	
27.	Methods of differentiation	×		
28.	Some applications of differentiation	×		
29.	Differentiation of parametric equations			
30.	Differentiation of implicit functions	×		
31.	Logarithmic differentiation	×		
32.	Differentiation of hyperbolic functions	×		
33.	Differentiation of inverse trigonometric and hyperbolic functions	×		
34.	Partial differentiation			×
35.	Total differential, rates of change and small changes			×
36.	Maxima, minima and saddle points for functions of two variables			×
37.	Standard integration	×		
38.	Some applications of integration	×		
39.	Integration using algebraic substitutions	×		
40.	Integration using trigonometric and hyperbolic substitutions	×		
41.	Integration using partial fractions	×		
42.	The $t=\tan\theta/2$ substitution			
43.	Integration by parts	×		
44.	Reduction formulae	×		
45.	Numerical integration		×	
46.	Solution of first order differential equations by separation of variables		×	
47.	Homogeneous first order differential equations			
48.	Linear first order differential equations		×	
49.	Numerical methods for first order differential equations		×	×
50.	Second order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$		×	
51.	Second order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$		×	
52.	Power series methods of solving ordinary differential equations			×
53.	An introduction to partial differential equations			×
54.	Presentation of statistical data	×		

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Engineering Mathematics
55.	Measures of central tendency and dispersion	×		
56.	Probability	×		
57.	The binomial and Poisson distributions	×		
58.	The normal distribution	×		
59.	Linear correlation	×		
60.	Linear regression	×		
61.	Introduction to Laplace transforms			×
62.	Properties of Laplace transforms			×
63.	Inverse Laplace transforms			×
64.	Solution of differential equations using Laplace transforms			×
65.	The solution of simultaneous differential equations using Laplace transforms			×
66.	Fourier series for periodic functions of period 2π			×
67.	Fourier series for non-periodic functions over range 2π			×
68.	Even and odd functions and half-range Fourier series			×
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Website Chapters				
72.	Inequalities			
73.	Boolean algebra and logic circuits		×	
74.	Sampling and estimation theories	×		
75.	Significance testing	×		
76.	Chi-square and distribution-free tests	×		

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Chapter 1

Algebra

1.1 Introduction

In this chapter, polynomial division and the factor and remainder theorems are explained (in Sections 1.4 to 1.6). However, before this, some essential algebra revision on basic laws and equations is included.

For further Algebra revision, go to website: http://books.elsevier.com/companions/0750681527

1.2 Revision of basic laws

(a) Basic operations and laws of indices

The **laws of indices** are:

(i) $a^m \times a^n = a^{m+n}$ (ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{m\times n}$ (iv) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$ (vi) $a^0 = 1$

Problem 1. Evaluate $4a^2bc^3 - 2ac$ when $a=2$, $b=\frac{1}{2}$ and $c=1\frac{1}{2}$

$$4a^2bc^3 - 2ac = 4(2)^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3 - 2(2)\left(\frac{3}{2}\right)$$

$$= \frac{4\times2\times2\times3\times3\times3}{2\times2\times2\times2} - \frac{12}{2}$$

$$= 27 - 6 = \mathbf{21}$$

Problem 2. Multiply $3x+2y$ by $x-y$.

$$\begin{array}{lr} & 3x + 2y \\ & x - y \\ \hline \text{Multiply by } x \rightarrow & 3x^2 + 2xy \\ \text{Multiply by } -y \rightarrow & -3xy - 2y^2 \\ \hline \text{Adding gives:} & \mathbf{3x^2 - xy - 2y^2} \\ \hline \end{array}$$

Alternatively,

$$(3x+2y)(x-y) = 3x^2 - 3xy + 2xy - 2y^2$$

$$= \mathbf{3x^2 - xy - 2y^2}$$

Problem 3. Simplify $\dfrac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a=3$, $b=\frac{1}{8}$ and $c=2$.

$$\frac{a^3b^2c^4}{abc^{-2}} = a^{3-1}b^{2-1}c^{4-(-2)} = \mathbf{a^2bc^6}$$

When $a=3$, $b=\frac{1}{8}$ and $c=2$,

$$a^2bc^6 = (3)^2\left(\frac{1}{8}\right)(2)^6 = (9)\left(\frac{1}{8}\right)(64) = \mathbf{72}$$

Problem 4. Simplify $\dfrac{x^2y^3 + xy^2}{xy}$

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy}$$

$$= x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$

$$= \mathbf{xy^2 + y} \quad \text{or} \quad \mathbf{y(xy+1)}$$

Problem 5. Simplify $\dfrac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}}$

$$\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}} = \frac{x^2y^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{5}{2}}y^{\frac{3}{2}}}$$

$$= x^{2+\frac{1}{2}-\frac{5}{2}}y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}$$

$$= x^0y^{-\frac{1}{3}}$$

$$= \mathbf{y^{-\frac{1}{3}}} \text{ or } \mathbf{\frac{1}{y^{\frac{1}{3}}}} \text{ or } \mathbf{\frac{1}{\sqrt[3]{y}}}$$

Now try the following exercise

Exercise 1 Revision of basic operations and laws of indices

1. Evaluate $2ab+3bc-abc$ when $a=2$, $b=-2$ and $c=4$. $[-16]$

2. Find the value of $5pq^2r^3$ when $p=\frac{2}{5}$, $q=-2$ and $r=-1$. $[-8]$

3. From $4x-3y+2z$ subtract $x+2y-3z$. $[3x-5y+5z]$

4. Multiply $2a-5b+c$ by $3a+b$. $[6a^2-13ab+3ac-5b^2+bc]$

5. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x=\frac{1}{2}$, $y=2$ and $z=3$. $[x^5y^4z^3, 13\frac{1}{2}]$

6. Evaluate $(a^{\frac{3}{2}}bc^{-3})(a^{\frac{1}{2}}b^{-\frac{1}{2}}c)$ when $a=3$, $b=4$ and $c=2$. $[\pm 4\frac{1}{2}]$

7. Simplify $\dfrac{a^2b+a^3b}{a^2b^2}$ $\left[\dfrac{1+a}{b}\right]$

8. Simplify $\dfrac{(a^3b^{\frac{1}{2}}c^{-\frac{1}{2}})(ab)^{\frac{1}{3}}}{(\sqrt{a^3}\sqrt{b}\,c)}$ $\left[a^{\frac{11}{6}}b^{\frac{1}{3}}c^{-\frac{3}{2}} \text{ or } \dfrac{\sqrt[6]{a^{11}}\sqrt[3]{b}}{\sqrt{c^3}}\right]$

(b) Brackets, factorization and precedence

Problem 6. Simplify $a^2-(2a-ab)-a(3b+a)$.

$$a^2-(2a-ab)-a(3b+a)$$
$$= a^2-2a+ab-3ab-a^2$$
$$= \mathbf{-2a-2ab} \text{ or } \mathbf{-2a(1+b)}$$

Problem 7. Remove the brackets and simplify the expression:

$$2a-[3\{2(4a-b)-5(a+2b)\}+4a].$$

Removing the innermost brackets gives:

$$2a-[3\{8a-2b-5a-10b\}+4a]$$

Collecting together similar terms gives:

$$2a-[3\{3a-12b\}+4a]$$

Removing the 'curly' brackets gives:

$$2a-[9a-36b+4a]$$

Collecting together similar terms gives:

$$2a-[13a-36b]$$

Removing the square brackets gives:

$$2a-13a+36b = \mathbf{-11a+36b} \text{ or } \mathbf{36b-11a}$$

Problem 8. Factorize (a) $xy-3xz$ (b) $4a^2+16ab^3$ (c) $3a^2b-6ab^2+15ab$.

(a) $xy-3xz = \mathbf{x(y-3z)}$

(b) $4a^2+16ab^3 = \mathbf{4a(a+4b^3)}$

(c) $3a^2b-6ab^2+15ab = \mathbf{3ab(a-2b+5)}$

Problem 9. Simplify $3c+2c\times 4c+c\div 5c-8c$.

The order of precedence is division, multiplication, addition and subtraction (sometimes remembered by BODMAS). Hence

$3c + 2c \times 4c + c \div 5c - 8c$

$= 3c + 2c \times 4c + \left(\frac{c}{5c}\right) - 8c$

$= 3c + 8c^2 + \frac{1}{5} - 8c$

$= \mathbf{8c^2 - 5c + \frac{1}{5}}$ or $\mathbf{c(8c - 5) + \frac{1}{5}}$

Problem 10. Simplify $(2a - 3) \div 4a + 5 \times 6 - 3a$.

$(2a - 3) \div 4a + 5 \times 6 - 3a$

$= \frac{2a-3}{4a} + 5 \times 6 - 3a$

$= \frac{2a-3}{4a} + 30 - 3a$

$= \frac{2a}{4a} - \frac{3}{4a} + 30 - 3a$

$= \frac{1}{2} - \frac{3}{4a} + 30 - 3a = \mathbf{30\frac{1}{2} - \frac{3}{4a} - 3a}$

Now try the following exercise

Exercise 2 Further problems on brackets, factorization and precedence

1. Simplify $2(p + 3q - r) - 4(r - q + 2p) + p$. $[-5p + 10q - 6r]$
2. Expand and simplify $(x + y)(x - 2y)$. $[x^2 - xy - 2y^2]$
3. Remove the brackets and simplify:

   $24p - [2\{3(5p - q) - 2(p + 2q)\} + 3q]$. $[11q - 2p]$
4. Factorize $21a^2b^2 - 28ab$. $[7ab(3ab - 4)]$
5. Factorize $2xy^2 + 6x^2y + 8x^3y$. $[2xy(y + 3x + 4x^2)]$
6. Simplify $2y + 4 \div 6y + 3 \times 4 - 5y$. $\left[\frac{2}{3y} - 3y + 12\right]$
7. Simplify $3 \div y + 2 \div y - 1$. $\left[\frac{5}{y} - 1\right]$
8. Simplify $a^2 - 3ab \times 2a \div 6b + ab$. $[ab]$

1.3 Revision of equations

(a) Simple equations

Problem 11. Solve $4 - 3x = 2x - 11$.

Since $4 - 3x = 2x - 11$ then $4 + 11 = 2x + 3x$

i.e. $15 = 5x$ from which, $\boldsymbol{x} = \frac{15}{5} = \mathbf{3}$

Problem 12. Solve $4(2a - 3) - 2(a - 4) = 3(a - 3) - 1$.

Removing the brackets gives:

$$8a - 12 - 2a + 8 = 3a - 9 - 1$$

Rearranging gives:

$8a - 2a - 3a = -9 - 1 + 12 - 8$

i.e. $3a = -6$

and $\boldsymbol{a} = \frac{-6}{3} = \mathbf{-2}$

Problem 13. Solve $\frac{3}{x-2} = \frac{4}{3x+4}$.

By 'cross-multiplying': $3(3x + 4) = 4(x - 2)$

Removing brackets gives: $9x + 12 = 4x - 8$

Rearranging gives: $9x - 4x = -8 - 12$

i.e. $5x = -20$

and $\boldsymbol{x} = \frac{-20}{5} = \mathbf{-4}$

Problem 14. Solve $\left(\frac{\sqrt{t} + 3}{\sqrt{t}}\right) = 2$.

$$\sqrt{t}\left(\frac{\sqrt{t}+3}{\sqrt{t}}\right)=2\sqrt{t}$$

i.e. $\sqrt{t}+3=2\sqrt{t}$

and $3=2\sqrt{t}-\sqrt{t}$

i.e. $3=\sqrt{t}$

and $\mathbf{9=t}$

(b) Transposition of formulae

Problem 15. Transpose the formula $v=u+\dfrac{ft}{m}$ to make f the subject.

$u+\dfrac{ft}{m}=v$ from which, $\dfrac{ft}{m}=v-u$

and $m\left(\dfrac{ft}{m}\right)=m(v-u)$

i.e. $ft=m(v-u)$

and $\boldsymbol{f=\dfrac{m}{t}(v-u)}$

Problem 16. The impedance of an a.c. circuit is given by $Z=\sqrt{R^2+X^2}$. Make the reactance X the subject.

$\sqrt{R^2+X^2}=Z$ and squaring both sides gives

$R^2+X^2=Z^2$, from which,

$X^2=Z^2-R^2$ and **reactance** $\boldsymbol{X=\sqrt{Z^2-R^2}}$

Problem 17. Given that $\dfrac{D}{d}=\sqrt{\left(\dfrac{f+p}{f-p}\right)}$, express p in terms of D, d and f.

Rearranging gives: $\sqrt{\left(\dfrac{f+p}{f-p}\right)}=\dfrac{D}{d}$

Squaring both sides gives: $\dfrac{f+p}{f-p}=\dfrac{D^2}{d^2}$

'Cross-multiplying' gives:

$$d^2(f+p)=D^2(f-p)$$

Removing brackets gives:

$$d^2f+d^2p=D^2f-D^2p$$

Rearranging gives: $d^2p+D^2p=D^2f-d^2f$

Factorizing gives: $p(d^2+D^2)=f(D^2-d^2)$

and $\boldsymbol{p=\dfrac{f(D^2-d^2)}{(d^2+D^2)}}$

Now try the following exercise

Exercise 3 Further problems on simple equations and transposition of formulae

In problems 1 to 4 solve the equations

1. $3x-2-5x=2x-4$. $\left[\frac{1}{2}\right]$
2. $8+4(x-1)-5(x-3)=2(5-2x)$. $[-3]$
3. $\dfrac{1}{3a-2}+\dfrac{1}{5a+3}=0$. $\left[-\frac{1}{8}\right]$
4. $\dfrac{3\sqrt{t}}{1-\sqrt{t}}=-6$. $[4]$
5. Transpose $y=\dfrac{3(F-f)}{L}$. for f.
   $\left[f=\dfrac{3F-yL}{3}\ \text{or}\ f=F-\dfrac{yL}{3}\right]$
6. Make l the subject of $t=2\pi\sqrt{\dfrac{l}{g}}$.
   $\left[l=\dfrac{t^2g}{4\pi^2}\right]$
7. Transpose $m=\dfrac{\mu L}{L+rCR}$ for L.
   $\left[L=\dfrac{mrCR}{\mu-m}\right]$
8. Make r the subject of the formula $\dfrac{x}{y}=\dfrac{1+r^2}{1-r^2}$. $\left[r=\sqrt{\left(\dfrac{x-y}{x+y}\right)}\right]$

(c) Simultaneous equations

Problem 18. Solve the simultaneous equations:

$$7x-2y=26 \quad (1)$$

$$6x+5y=29. \quad (2)$$

$5 \times$ equation (1) gives:

$$35x - 10y = 130 \qquad (3)$$

$2 \times$ equation (2) gives:

$$12x + 10y = 58 \qquad (4)$$

equation (3) + equation (4) gives:

$$47x + 0 = 188$$

from which, $\boldsymbol{x} = \dfrac{188}{47} = \mathbf{4}$

Substituting $x = 4$ in equation (1) gives:

$$28 - 2y = 26$$

from which, $28 - 26 = 2y$ and $\boldsymbol{y} = \mathbf{1}$

Problem 19. Solve

$$\frac{x}{8} + \frac{5}{2} = y \qquad (1)$$

$$11 + \frac{y}{3} = 3x. \qquad (2)$$

$8 \times$ equation (1) gives: $x + 20 = 8y$ (3)

$3 \times$ equation (2) gives: $33 + y = 9x$ (4)

i.e. $x - 8y = -20$ (5)

and $9x - y = 33$ (6)

$8 \times$ equation (6) gives: $72x - 8y = 264$ (7)

Equation (7) − equation (5) gives:

$$71x = 284$$

from which, $\boldsymbol{x} = \dfrac{284}{71} = \mathbf{4}$

Substituting $x = 4$ in equation (5) gives:

$$4 - 8y = -20$$

from which, $4 + 20 = 8y$ and $\boldsymbol{y} = \mathbf{3}$

(d) Quadratic equations

Problem 20. Solve the following equations by factorization:
(a) $3x^2 - 11x - 4 = 0$
(b) $4x^2 + 8x + 3 = 0$.

(a) The factors of $3x^2$ are $3x$ and x and these are placed in brackets thus:

$(3x \quad)(x \quad)$

The factors of -4 are $+1$ and -4 or -1 and $+4$, or -2 and $+2$. Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.,

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

Thus $(3x + 1)(x - 4) = 0$ hence

either $(3x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $(x - 4) = 0$ i.e. $\boldsymbol{x = 4}$

(b) $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Thus $(2x + 3)(2x + 1) = 0$ hence

either $(2x + 3) = 0$ i.e. $\boldsymbol{x = -\frac{3}{2}}$

or $(2x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{2}}$

Problem 21. The roots of a quadratic equation are $\frac{1}{3}$ and -2. Determine the equation in x.

If $\frac{1}{3}$ and -2 are the roots of a quadratic equation then,

$$(x - \tfrac{1}{3})(x + 2) = 0$$

i.e. $x^2 + 2x - \frac{1}{3}x - \frac{2}{3} = 0$

i.e. $x^2 + \frac{5}{3}x - \frac{2}{3} = 0$

or $\boldsymbol{3x^2 + 5x - 2 = 0}$

Problem 22. Solve $4x^2 + 7x + 2 = 0$ giving the answer correct to 2 decimal places.

From the quadratic formula if $ax^2 + bx + c = 0$ then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence if $4x^2 + 7x + 2 = 0$

then $x = \dfrac{-7 \pm \sqrt{7^2 - 4(4)(2)}}{2(4)}$

$$= \frac{-7 \pm \sqrt{17}}{8}$$

$$= \frac{-7 \pm 4.123}{8}$$

$$= \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

i.e. $\boldsymbol{x = -0.36}$ or $\boldsymbol{-1.39}$

Now try the following exercise

Exercise 4 Further problems on simultaneous and quadratic equations

In problems 1 to 3, solve the simultaneous equations

1. $8x - 3y = 51$

   $3x + 4y = 14.$ $[x = 6,\ y = -1]$

2. $5a = 1 - 3b$

   $2b + a + 4 = 0.$ $[a = 2,\ b = -3]$

3. $\frac{x}{5} + \frac{2y}{3} = \frac{49}{15}$

   $\frac{3x}{7} - \frac{y}{2} + \frac{5}{7} = 0.$ $[x = 3,\ y = 4]$

4. Solve the following quadratic equations by factorization:

   (a) $x^2 + 4x - 32 = 0$

   (b) $8x^2 + 2x - 15 = 0.$

   [(a) 4, −8 (b) $\frac{5}{4}, -\frac{3}{2}$]

5. Determine the quadratic equation in x whose roots are 2 and −5.

   $[x^2 + 3x - 10 = 0]$

6. Solve the following quadratic equations, correct to 3 decimal places:

   (a) $2x^2 + 5x - 4 = 0$

   (b) $4t^2 - 11t + 3 = 0.$

   $\begin{bmatrix} \text{(a) } 0.637, -3.137 \\ \text{(b) } 2.443, 0.307 \end{bmatrix}$

1.4 Polynomial division

Before looking at long division in algebra let us revise long division with numbers (we may have forgotten, since calculators do the job for us!)

For example, $\frac{208}{16}$ is achieved as follows:

```
       13
   ______
16 ) 208
     16
     __
      48
      48
      __
      ..
      __
```

(1) 16 divided into 2 won't go

(2) 16 divided into 20 goes 1

(3) Put 1 above the zero

(4) Multiply 16 by 1 giving 16

(5) Subtract 16 from 20 giving 4

(6) Bring down the 8

(7) 16 divided into 48 goes 3 times

(8) Put the 3 above the 8

(9) $3 \times 16 = 48$

(10) $48 - 48 = 0$

Hence $\frac{208}{16} = \mathbf{13}$ exactly

Similarly, $\frac{172}{15}$ is laid out as follows:

```
       11
   ______
15 ) 172
     15
     __
      22
      15
      __
       7
      __
```

Hence $\frac{172}{15} = 11$ remainder 7 or $11 + \frac{7}{15} = \mathbf{11\frac{7}{15}}$

Below are some examples of division in algebra, which in some respects, is similar to long division with numbers.

(Note that a **polynomial** is an expression of the form

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$

and **polynomial division** is sometimes required when resolving into partial fractions—see Chapter 2.)

Problem 23. Divide $2x^2 + x - 3$ by $x - 1$.

$2x^2 + x - 3$ is called the **dividend** and $x - 1$ the **divisor**. The usual layout is shown below with the dividend and divisor both arranged in descending powers of the symbols.

$$\begin{array}{r|l} & 2x + 3 \\ \hline x - 1 & 2x^2 + x - 3 \\ & 2x^2 - 2x \\ \hline & \quad 3x - 3 \\ & \quad 3x - 3 \\ \hline & \quad \cdot \quad \cdot \end{array}$$

Dividing the first term of the dividend by the first term of the divisor, i.e. $\frac{2x^2}{x}$ gives $2x$, which is put above the first term of the dividend as shown. The divisor is then multiplied by $2x$, i.e. $2x(x - 1) = 2x^2 - 2x$, which is placed under the dividend as shown. Subtracting gives $3x - 3$. The process is then repeated, i.e. the first term of the divisor, x, is divided into $3x$, giving $+3$, which is placed above the dividend as shown. Then $3(x - 1) = 3x - 3$ which is placed under the $3x - 3$. The remainder, on subtraction, is zero, which completes the process.

Thus $\mathbf{(2x^2 + x - 3) \div (x - 1) = (2x + 3)}$

[A check can be made on this answer by multiplying $(2x + 3)$ by $(x - 1)$ which equals $2x^2 + x - 3$]

Problem 24. Divide $3x^3 + x^2 + 3x + 5$ by $x + 1$.

$$\begin{array}{r|l} & (1)\ \ (4)\ \ (7) \\ & 3x^2 - 2x \ \ + 5 \\ \hline x + 1 & 3x^3 + x^2 + 3x + 5 \\ & 3x^3 + 3x^2 \\ \hline & \quad -2x^2 + 3x + 5 \\ & \quad -2x^2 - 2x \\ \hline & \qquad\quad 5x + 5 \\ & \qquad\quad 5x + 5 \\ \hline & \qquad\quad \cdot \quad \cdot \end{array}$$

(1) x into $3x^3$ goes $3x^2$. Put $3x^2$ above $3x^3$

(2) $3x^2(x + 1) = 3x^3 + 3x^2$

(3) Subtract

(4) x into $-2x^2$ goes $-2x$. Put $-2x$ above the dividend

(5) $-2x(x + 1) = -2x^2 - 2x$

(6) Subtract

(7) x into $5x$ goes 5. Put 5 above the dividend

(8) $5(x + 1) = 5x + 5$

(9) Subtract

Thus $$\frac{3x^3 + x^2 + 3x + 5}{x + 1} = \mathbf{3x^2 - 2x + 5}$$

Problem 25. Simplify $\frac{x^3 + y^3}{x + y}$.

$$\begin{array}{r|l} & (1)\ \ (4)\ \ (7) \\ & x^2 - xy \ + y^2 \\ \hline x + y & x^3 + 0 + 0 + y^3 \\ & x^3 + x^2y \\ \hline & \quad -x^2y \qquad + y^3 \\ & \quad -x^2y - xy^2 \\ \hline & \qquad\qquad xy^2 + y^3 \\ & \qquad\qquad xy^2 + y^3 \\ \hline & \qquad\qquad \cdot \quad \cdot \end{array}$$

(1) x into x^3 goes x^2. Put x^2 above x^3 of dividend

(2) $x^2(x + y) = x^3 + x^2y$

(3) Subtract

(4) x into $-x^2y$ goes $-xy$. Put $-xy$ above dividend

(5) $-xy(x + y) = -x^2y - xy^2$

(6) Subtract

(7) x into xy^2 goes y^2. Put y^2 above dividend

(8) $y^2(x + y) = xy^2 + y^3$

(9) Subtract

Thus

$$\frac{x^3 + y^3}{x + y} = \mathbf{x^2 - xy + y^2}$$

The zero's shown in the dividend are not normally shown, but are included to clarify the subtraction process and to keep similar terms in their respective columns.

Problem 26. Divide (x^2+3x-2) by $(x-2)$.

$$\begin{array}{r|l} & x \quad +5 \\ \hline x-2 & x^2+3x-\ \ 2 \\ & x^2-2x \\ \hline & \qquad 5x-\ \ 2 \\ & \qquad 5x-10 \\ \hline & \qquad\qquad 8 \\ \hline \end{array}$$

Hence

$$\frac{x^2+3x-2}{x-2}=\mathbf{x+5+\frac{8}{x-2}}$$

Problem 27. Divide $4a^3-6a^2b+5b^3$ by $2a-b$.

$$\begin{array}{r|l} & 2a^2-2ab-\ \ b^2 \\ \hline 2a-b & 4a^3-6a^2b \qquad\quad +5b^3 \\ & 4a^3-2a^2b \\ \hline & \quad -4a^2b \qquad\quad +5b^3 \\ & \quad -4a^2b+2ab^2 \\ \hline & \qquad\qquad -2ab^2+5b^3 \\ & \qquad\qquad -2ab^2+\ b^3 \\ \hline & \qquad\qquad\qquad 4b^3 \\ \hline \end{array}$$

Thus

$$\frac{4a^3-6a^2b+5b^3}{2a-b}$$

$$=\mathbf{2a^2-2ab-b^2+\frac{4b^3}{2a-b}}$$

Now try the following exercise

Exercise 5 Further problems on polynomial division

1. Divide $(2x^2+xy-y^2)$ by $(x+y)$. $[2x-y]$
2. Divide $(3x^2+5x-2)$ by $(x+2)$. $[3x-1]$
3. Determine $(10x^2+11x-6)\div(2x+3)$. $[5x-2]$
4. Find $\dfrac{14x^2-19x-3}{2x-3}$. $[7x+1]$
5. Divide $(x^3+3x^2y+3xy^2+y^3)$ by $(x+y)$. $[x^2+2xy+y^2]$
6. Find $(5x^2-x+4)\div(x-1)$. $\left[5x+4+\dfrac{8}{x-1}\right]$
7. Divide $(3x^3+2x^2-5x+4)$ by $(x+2)$. $\left[3x^2-4x+3-\dfrac{2}{x+2}\right]$
8. Determine $(5x^4+3x^3-2x+1)/(x-3)$. $\left[5x^3+18x^2+54x+160+\dfrac{481}{x-3}\right]$

1.5 The factor theorem

There is a simple relationship between the factors of a quadratic expression and the roots of the equation obtained by equating the expression to zero.
For example, consider the quadratic equation $x^2+2x-8=0$.
To solve this we may factorize the quadratic expression x^2+2x-8 giving $(x-2)(x+4)$.
Hence $(x-2)(x+4)=0$.
Then, if the product of two numbers is zero, one or both of those numbers must equal zero. Therefore,

either $(x-2)=0$, from which, $x=2$

or $(x+4)=0$, from which, $x=-4$

It is clear then that a factor of $(x-2)$ indicates a root of +2, while a factor of $(x+4)$ indicates a root of −4.

In general, we can therefore say that:

a factor of $(x-a)$ corresponds to a root of $x=a$

In practice, we always deduce the roots of a simple quadratic equation from the factors of the quadratic expression, as in the above example. However, we could reverse this process. If, by trial and error, we could determine that $x=2$ is a root of the equation $x^2+2x-8=0$ we could deduce at once that $(x-2)$ is a factor of the expression x^2+2x-8. We wouldn't normally solve quadratic equations this way — but suppose we have to factorize a cubic expression (i.e. one in which the highest power of the variable is 3). A cubic equation might have three simple linear factors and the difficulty of discovering all these factors by trial and error would be considerable. It is to deal with this kind of case that we use the **factor theorem**. This is just a generalized version of what we established above for the quadratic expression. The factor theorem provides a method of factorizing any polynomial, $f(x)$, which has simple factors.

A statement of the **factor theorem** says:

'if $x = a$ is a root of the equation
$f(x) = 0$, then $(x - a)$ is a factor of $f(x)$'

The following worked problems show the use of the factor theorem.

Problem 28. Factorize x^3-7x-6 and use it to solve the cubic equation $x^3-7x-6=0$.

Let $f(x)=x^3-7x-6$

If $x=1$, then $f(1)=1^3-7(1)-6=-12$

If $x=2$, then $f(2)=2^3-7(2)-6=-12$

If $x=3$, then $f(3)=3^3-7(3)-6=0$

If $f(3)=0$, then $(x-3)$ is a factor — from the factor theorem.
We have a choice now. We can divide x^3-7x-6 by $(x-3)$ or we could continue our 'trial and error' by substituting further values for x in the given expression — and hope to arrive at $f(x)=0$.
Let us do both ways. Firstly, dividing out gives:

$$\begin{array}{r|l} & x^2+3x+2 \\ \hline x-3 & x^3-0\quad -7x-6 \\ & x^3-3x^2 \\ \hline & \quad 3x^2-7x-6 \\ & \quad 3x^2-9x \\ \hline & \qquad 2x-6 \\ & \qquad 2x-6 \\ \hline & \qquad \cdot \quad \cdot \end{array}$$

Hence $\dfrac{x^3-7x-6}{x-3}=x^2+3x+2$

i.e. $x^3-7x-6=(x-3)(x^2+3x+2)$

x^2+3x+2 factorizes 'on sight' as $(x+1)(x+2)$. Therefore

$$\mathbf{x^3-7x-6=(x-3)(x+1)(x+2)}$$

A second method is to continue to substitute values of x into $f(x)$.

Our expression for $f(3)$ was $3^3-7(3)-6$. We can see that if we continue with positive values of x the first term will predominate such that $f(x)$ will not be zero.

Therefore let us try some negative values for x. Therefore $f(-1)=(-1)^3-7(-1)-6=0$; hence $(x+1)$ is a factor (as shown above). Also $f(-2)=(-2)^3-7(-2)-6=0$; hence $(x+2)$ is a factor (also as shown above).

To solve $x^3-7x-6=0$, we substitute the factors, i.e.,

$$(x-3)(x+1)(x+2)=0$$

from which, $\mathbf{x=3, x=-1}$ **and** $\mathbf{x=-2}$.

Note that the values of x, i.e. 3, -1 and -2, are all factors of the constant term, i.e. the 6. This can give us a clue as to what values of x we should consider.

Problem 29. Solve the cubic equation $x^3-2x^2-5x+6=0$ by using the factor theorem.

Let $f(x)=x^3-2x^2-5x+6$ and let us substitute simple values of x like 1, 2, 3, -1, -2, and so on.

$f(1)=1^3-2(1)^2-5(1)+6=0$,
hence $(x-1)$ is a factor

$f(2)=2^3-2(2)^2-5(2)+6\neq 0$

$f(3)=3^3-2(3)^2-5(3)+6=0$,
hence $(x-3)$ is a factor

$f(-1)=(-1)^3-2(-1)^2-5(-1)+6\neq 0$

$f(-2)=(-2)^3-2(-2)^2-5(-2)+6=0$,
hence $(x+2)$ is a factor

Hence $x^3-2x^2-5x+6=(x-1)(x-3)(x+2)$
Therefore if $x^3-2x^2-5x+6=0$
then $(x-1)(x-3)(x+2)=0$
from which, $\mathbf{x=1, x=3}$ **and** $\mathbf{x=-2}$

Alternatively, having obtained one factor, i.e. $(x-1)$ we could divide this into (x^3-2x^2-5x+6) as follows:

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\ \hline
x-1 & x^3 - 2x^2 - 5x + 6 \\
 & x^3 - x^2 \\ \hline
 & \quad - x^2 - 5x + 6 \\
 & \quad - x^2 + x \\ \hline
 & \qquad - 6x + 6 \\
 & \qquad - 6x + 6 \\ \hline
 & \qquad \cdot \quad \cdot
\end{array}
$$

Hence $x^3 - 2x^2 - 5x + 6$

$= (x-1)(x^2 - x - 6)$

$= \mathbf{(x-1)(x-3)(x+2)}$

Summarizing, the factor theorem provides us with a method of factorizing simple expressions, and an alternative, in certain circumstances, to polynomial division.

Now try the following exercise

Exercise 6 Further problems on the factor theorem

Use the factor theorem to factorize the expressions given in problems 1 to 4.

1. $x^2 + 2x - 3$ $\quad [(x-1)(x+3)]$

2. $x^3 + x^2 - 4x - 4$ $\quad [(x+1)(x+2)(x-2)]$

3. $2x^3 + 5x^2 - 4x - 7$ $\quad [(x+1)(2x^2 + 3x - 7)]$

4. $2x^3 - x^2 - 16x + 15$ $\quad [(x-1)(x+3)(2x-5)]$

5. Use the factor theorem to factorize $x^3 + 4x^2 + x - 6$ and hence solve the cubic equation $x^3 + 4x^2 + x - 6 = 0$.
$\left[\begin{array}{l} x^3 + 4x^2 + x - 6 \\ \quad = (x-1)(x+3)(x+2) \\ x = 1, x = -3 \text{ and } x = -2 \end{array}\right]$

6. Solve the equation $x^3 - 2x^2 - x + 2 = 0$.
$[x = 1, x = 2 \text{ and } x = -1]$

1.6 The remainder theorem

Dividing a general quadratic expression $(ax^2 + bx + c)$ by $(x - p)$, where p is any whole number, by long division (see section 1.3) gives:

$$
\begin{array}{r|l}
 & ax \ + (b + ap) \\ \hline
x-p & ax^2 + bx \qquad\quad + c \\
 & ax^2 - apx \\ \hline
 & \qquad (b+ap)x + c \\
 & \qquad (b+ap)x - (b+ap)p \\ \hline
 & \qquad\qquad c + (b+ap)p \\ \hline
\end{array}
$$

The remainder, $c + (b+ap)p = c + bp + ap^2$ or $ap^2 + bp + c$. This is, in fact, what the **remainder theorem** states, i.e.,

'if $(ax^2 + bx + c)$ is divided by $(x - p)$, the remainder will be $ap^2 + bp + c$'

If, in the dividend $(ax^2 + bx + c)$, we substitute p for x we get the remainder $ap^2 + bp + c$.

For example, when $(3x^2 - 4x + 5)$ is divided by $(x - 2)$ the remainder is $ap^2 + bp + c$ (where $a = 3$, $b = -4$, $c = 5$ and $p = 2$),
i.e. the remainder is

$$3(2)^2 + (-4)(2) + 5 = 12 - 8 + 5 = \mathbf{9}$$

We can check this by dividing $(3x^2 - 4x + 5)$ by $(x - 2)$ by long division:

$$
\begin{array}{r|l}
 & 3x + 2 \\ \hline
x-2 & 3x^2 - 4x + 5 \\
 & 3x^2 - 6x \\ \hline
 & \qquad 2x + 5 \\
 & \qquad 2x - 4 \\ \hline
 & \qquad\quad 9 \\ \hline
\end{array}
$$

Similarly, when $(4x^2 - 7x + 9)$ is divided by $(x + 3)$, the remainder is $ap^2 + bp + c$, (where $a = 4$, $b = -7$, $c = 9$ and $p = -3$) i.e. the remainder is $4(-3)^2 + (-7)(-3) + 9 = 36 + 21 + 9 = \mathbf{66}$.

Also, when $(x^2 + 3x - 2)$ is divided by $(x - 1)$, the remainder is $1(1)^2 + 3(1) - 2 = \mathbf{2}$.

It is not particularly useful, on its own, to know the remainder of an algebraic division. However, if the remainder should be zero then $(x - p)$ is a factor. This is very useful therefore when factorizing expressions.

For example, when $(2x^2 + x - 3)$ is divided by $(x - 1)$, the remainder is $2(1)^2 + 1(1) - 3 = 0$, which means that $(x - 1)$ is a factor of $(2x^2 + x - 3)$.