Arthur Engel

# Problem-Solving Strategies

With 223 Figures

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# Preface

This book is an outgrowth of the training of the German IMO team from a time when we had only a short training time of 14 days, including 6 half-day tests. This has forced upon us a training of enormous compactness. “Great Ideas” were the leading principles. A huge number of problems were selected to illustrate these principles. Not only topics but also ideas were efficient means of classification.

For whom is this book written?

- For trainers and participants of contests of all kinds up to the highest level of international competitions, including the IMO and the Putnam Competition.
- For the regular high school teacher, who is conducting a mathematics club and is looking for ideas and problems for his/her club. Here, he/she will find problems of any level from very simple ones to the most difficult problems ever proposed at any competition.
- For high school teachers who want to pose *the problem of the week, problem of the month,* and *research problems of the year.* This is not so easy. Many fail, but some persevere, and after a while they succeed and generate a creative atmosphere with continuous discussions of mathematical problems.
- For the regular high school teacher, who is just looking for ideas to enrich his/her teaching by some interesting nonroutine problems.
- For all those who are interested in solving tough and interesting problems.

The book is organized into chapters. Each chapter starts with typical examples illustrating the main ideas followed by many problems and their solutions. The solutions are sometimes just hints, giving away the main idea leading to the solution. In this way, it was possible to increase the number of examples and problems to over 1300. The reader can increase the effectiveness of the book even more by trying to solve the examples.

The problems are almost exclusively competition problems from all over the world. Most of them are from the former USSR, some from Hungary, and some from Western countries, especially from the German National Competition. The competition problems are usually variations of problems from journals with problem sections. So it is not always easy to give credit to the originators of the problem. If you see a beautiful problem, you first wonder at the creativity of the problem proposer. Later you discover the result in an earlier source. For this reason, the references to competitions are somewhat sporadic. Usually no source is given if I have known the problem for more than 25 years. Anyway, most of the problems are results that are known to experts in the respective fields.

There is a huge literature of mathematical problems. But, as a trainer, I know that there can never be enough problems. You are always in desperate need of new problems or old problems with new solutions. Any new problem book has some new problems, and a big book, as this one, usually has quite a few problems that are new to the reader.

The problems are arranged in no particular order, and especially not in increasing order of difficulty. We do not know how to rate a problem's difficulty. Even the IMO jury, now consisting of 75 highly skilled problem solvers, commits grave errors in rating the difficulty of the problems it selects. The over 400 IMO contestants are also an unreliable guide. Too much depends on the previous training by an ever-changing set of hundreds of trainers. A problem changes from impossible to trivial if a related problem was solved in training.

I would like to thank Dr. Manfred Grathwohl for his help in implementing various LaTeX versions on the workstation at the institute and on my PC at home. When difficulties arose, he was a competent and friendly advisor.

There will be some errors in the proofs, for which I take full responsibility, since none of my colleagues has read the manuscript before. Readers will miss important strategies. So do I, but I have set myself a limit to the size of the book. Especially, advanced methods are missing. Still, it is probably the most complete training book on the market. The gravest gap is the absence of new topics like probability and algorithmics to counter the conservative mood of the IMO jury. One exception is Chapter 13 on games, a topic almost nonexistent in the IMO, but very popular in Russia.

Frankfurt am Main, Germany Arthur Engel

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# Abbreviations and Notations

## Abbreviations

| | |
|---|---|
| ARO | Allrussian Mathematical Olympiad |
| ATMO | Austrian Mathematical Olympiad |
| AuMO | Australian Mathematical Olympiad |
| AUO | Allunion Mathematical Olympiad |
| BrMO | British Mathematical Olympiad |
| BWM | German National Olympiad |
| BMO | Balkan Mathematical Olympiad |
| ChNO | Chinese National Olympiad |
| HMO | Hungarian Mathematical Olympiad (Kűrschak Competition) |
| IIM | International Intellectual Marathon (Mathematics/Physics Competition) |
| IMO | International Mathematical Olympiad |
| LMO | Leningrad Mathematical Olympiad |
| MMO | Moskov Mathematical Olympiad |
| PAMO | Polish-Austrian Mathematical Olympiad |

| | |
|---|---|
| PMO | Polish Mathematical Olympiad |
| RO | Russian Olympiad (ARO from 1994 on) |
| SPMO | St. Petersburg Mathematical Olympiad |
| TT | Tournament of the Towns |
| USO | US Olympiad |

## Notations for Numerical Sets

| | |
|---|---|
| $\mathbb{N}$ or $\mathbb{Z}^+$ | the positive integers (natural numbers), i.e., $\{1,2,3,\ldots\}$ |
| $\mathbb{N}_0$ | the nonnegative integers, $\{0,1,2,\ldots\}$ |
| $\mathbb{Z}$ | the integers |
| $\mathbb{Q}$ | the rational numbers |
| $\mathbb{Q}^+$ | the positive rational numbers |
| $\mathbb{Q}_0^+$ | the nonnegative rational numbers |
| $\mathbb{R}$ | the real numbers |
| $\mathbb{R}^+$ | the positive real numbers |
| $\mathbb{C}$ | the complex numbers |
| $\mathbb{Z}_n$ | the integers modulo $n$ |
| $1\,..\,n$ | the integers $1,\ 2,\ \ldots,\ n$ |

## Notations from Sets, Logic, and Geometry

| | |
|---|---|
| $\Longleftrightarrow$ | iff, if and only if |
| $\Longrightarrow$ | implies |
| $A \subset B$ | $A$ is a subset of $B$ |
| $A \setminus B$ | $A$ without $B$ |
| $A \cap B$ | the intersection of $A$ and $B$ |
| $A \cup B$ | the union of $A$ and $B$ |
| $a \in A$ | the element $a$ belongs to the set $A$ |
| $\lvert AB \rvert$ | also $AB$, the distance between the points $A$ and $B$ |
| box | parallelepiped, solid bounded by three pairs of parallel planes |

# 1

# The Invariance Principle

We present our first *Higher Problem-Solving Strategy*. It is extremely useful in solving certain types of difficult problems, which are easily recognizable. We will teach it by solving problems which use this strategy. In fact, **problem solving can be learned only by solving problems.** But it must be supported by strategies provided by the trainer.

Our first strategy is the *search for invariants*, and it is called the **Invariance Principle**. The principle is applicable to algorithms (games, transformations). Some task is repeatedly performed. **What stays the same? What remains invariant?** Here is a saying easy to remember:

**If there is repetition, look for what does not change!**

In algorithms there is a starting state $S$ and a sequence of legal steps (moves, transformations). One looks for answers to the following questions:

1. Can a given end state be reached?
2. Find all reachable end states.
3. Is there convergence to an end state?
4. Find all periods with or without tails, if any.

Since the Invariance Principle is a *heuristic principle*, it is best learned by experience, which we will gain by solving the key examples **E1** to **E10**.

**E1.** *Starting with a point $S = (a, b)$ of the plane with $0 < b < a$, we generate a sequence of points $(x_n, y_n)$ according to the rule*

$$x_0 = a, \quad y_0 = b, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}.$$

Here it is easy to find an *invariant*. From $x_{n+1}y_{n+1} = x_n y_n$, for all $n$ we deduce $x_n y_n = ab$ for all $n$. This is the *invariant* we are looking for. Initially, we have $y_0 < x_0$. This relation also remains invariant. Indeed, suppose $y_n < x_n$ for some $n$. Then $x_{n+1}$ is the midpoint of the segment with endpoints $y_n, x_n$. Moreover, $y_{n+1} < x_{n+1}$ since the harmonic mean is strictly less than the arithmetic mean. Thus,

$$0 < x_{n+1} - y_{n+1} = \frac{x_n - y_n}{x_n + y_n} \cdot \frac{x_n - y_n}{2} < \frac{x_n - y_n}{2}$$

for all $n$. So we have $\lim x_n = \lim y_n = x$ with $x^2 = ab$ or $x = \sqrt{ab}$.

Here the invariant helped us very much, but its recognition was not yet the solution, although the completion of the solution was trivial.

**E2.** *Suppose the positive integer n is odd. First Al writes the numbers* $1, 2, \ldots, 2n$ *on the blackboard. Then he picks any two numbers $a$, $b$, erases them, and writes, instead, $|a - b|$. Prove that an odd number will remain at the end.*

**Solution.** Suppose $S$ is the sum of all the numbers still on the blackboard. Initially this sum is $S = 1+2+\cdots+2n = n(2n+1)$, an odd number. Each step reduces $S$ by $2\min(a, b)$, which is an even number. So the parity of $S$ is an *invariant*. During the whole reduction process we have $S \equiv 1 \bmod 2$. Initially the parity is odd. So, it will also be odd at the end.

**E3.** *A circle is divided into six sectors. Then the numbers* $1, 0, 1, 0, 0, 0$ *are written into the sectors (counterclockwise, say). You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?*

**Solution.** Suppose $a_1, \ldots, a_6$ are the numbers currently on the sectors. Then $I = a_1 - a_2 + a_3 - a_4 + a_5 - a_6$ is an *invariant*. Initially $I = 2$. The goal $I = 0$ cannot be reached.

**E4.** *In the Parliament of Sikinia, each member has* **at most three enemies.** *Prove that the house can be separated into two houses, so that each member has* **at most one enemy** *in his own house.*

**Solution.** Initially, we separate the members in any way into the two houses. Let $H$ be the total sum of all the enemies each member has in his own house. Now suppose $A$ has at least two enemies in his own house. Then he has at most one enemy in the other house. If $A$ switches houses, the number $H$ will decrease. This decrease cannot go on forever. At some time, $H$ reaches its absolute minimum. Then we have reached the required distribution.

Here we have a new idea. We construct a positive integral function which decreases at each step of the algorithm. So we know that our algorithm will terminate. There is no strictly decreasing infinite sequence of positive integers. $H$ is not strictly an invariant, but decreases monotonically until it becomes constant. Here, the monotonicity relation is the invariant.

**E5.** *Suppose not all four integers $a, b, c, d$ are equal. Start with $(a, b, c, d)$ and repeatedly replace $(a, b, c, d)$ by $(a - b, b - c, c - d, d - a)$. Then at least one number of the quadruple will eventually become arbitrarily large.*

**Solution.** Let $P_n = (a_n, b_n, c_n, d_n)$ be the quadruple after $n$ iterations. Then we have $a_n + b_n + c_n + d_n = 0$ for $n \geq 1$. We do not see yet how to use this invariant. But geometric interpretation is mostly helpful. A very important function for the point $P_n$ in 4-space is the square of its distance from the origin (0, 0, 0, 0), which is $a_n^2 + b_n^2 + c_n^2 + d_n^2$. If we could prove that it has no upper bound, we would be finished.

We try to find a relation between $P_{n+1}$ and $P_n$:

$$\begin{aligned} a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2 &= (a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - d_n)^2 + (d_n - a_n)^2 \\ &= 2(a_n^2 + b_n^2 + c_n^2 + d_n^2) \\ &\quad - 2a_nb_n - 2b_nc_n - 2c_nd_n - 2d_na_n. \end{aligned}$$

Now we can use $a_n + b_n + c_n + d_n = 0$ or rather its square:

$$0 = (a_n+b_n+c_n+d_n)^2 = (a_n+c_n)^2+(b_n+d_n)^2+2a_nb_n+2a_nd_n+2b_nc_n+2c_nd_n. \tag{1}$$

Adding (1) and (2), for $a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2$, we get

$$2(a_n^2 + b_n^2 + c_n^2 + d_n^2) + (a_n + c_n)^2 + (b_n + d_n)^2 \geq 2(a_n^2 + b_n^2 + c_n^2 + d_n^2).$$

From this invariant inequality relationship we conclude that, for $n \geq 2$,

$$a_n^2 + b_n^2 + c_n^2 + d_n^2 \geq 2^{n-1}(a_1^2 + b_1^2 + c_1^2 + d_1^2). \tag{2}$$

The distance of the points $P_n$ from the origin increases without bound, which means that at least one component must become arbitrarily large. Can you always have equality in (2)?

**Here we learned that the distance from the origin is a very important function. Each time you have a sequence of points you should consider it.**

**E6.** *An algorithm is defined as follows:*

$$\textit{Start:}\quad (x_0, y_0) \text{ with } 0 < x_0 < y_0.$$

$$\textit{Step:}\quad x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_{n+1}y_n}.$$

Figure 1.1 and the arithmetic mean-geometric mean inequality show that

$$x_n < y_n \Rightarrow x_{n+1} < y_{n+1}, \quad y_{n+1} - x_{n+1} < \frac{y_n - x_n}{4}$$

for all $n$. Find the common limit $\lim x_n = \lim y_n = x = y$.

Here, invariants can help. But there are no systematic methods to find invariants, just *heuristics*. These are methods which often work, but not always. Two of these heuristics tell us to look for the change in $x_n/y_n$ or $y_n - x_n$ when going from $n$ to $n + 1$.

(a)
$$\frac{x_{n+1}}{y_{n+1}} = \frac{x_{n+1}}{\sqrt{x_{n+1}y_n}} = \sqrt{\frac{x_{n+1}}{y_n}} = \sqrt{\frac{1 + x_n/y_n}{2}}. \tag{1}$$

This reminds us of the half-angle relation

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{2}}.$$

Since we always have $0 < x_n/y_n < 1$, we may set $x_n/y_n = \cos\alpha_n$. Then (1) becomes

$$\cos\alpha_{n+1} = \cos\frac{\alpha_n}{2} \Rightarrow \alpha_n = \frac{\alpha_0}{2^n} \Rightarrow 2^n\alpha_n = \alpha_0,$$

which is equivalent to

$$2^n \arccos\frac{x_n}{y_n} = \arccos\frac{x_0}{y_0}. \tag{2}$$

This is an *invariant!*

(b) To avoid square roots, we consider $y_n^2 - x_n^2$ instead of $y_n - x_n$ and get

$$y_{n+1}^2 - x_{n+1}^2 = \frac{y_n^2 - x_n^2}{4} \Rightarrow 2\sqrt{y_{n+1}^2 - x_{n+1}^2} = \sqrt{y_n^2 - x_n^2}$$

or

$$2^n\sqrt{y_n^2 - x_n^2} = \sqrt{y_0^2 - x_0^2}, \tag{3}$$

which is a second *invariant*.

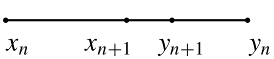


Fig. 1.1

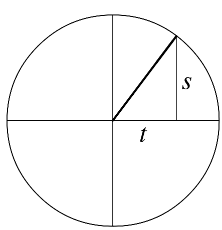


Fig. 1.2. $\arccos t = \arcsin s, \ s = \sqrt{1 - t^2}$.

From Fig. 1.2 and (2), (3), we get

$$\arccos\frac{x_0}{y_0} = 2^n \arccos\frac{x_n}{y_n} = 2^n \arcsin\frac{\sqrt{y_n^2 - x_n^2}}{y_n} = 2^n \arcsin\frac{\sqrt{y_0^2 - x_0^2}}{2^n y_n}.$$

The right-hand side converges to $\sqrt{y_0^2 - x_0^2}/y$ for $n \to \infty$. Finally, we get

$$x = y = \frac{\sqrt{y_0^2 - x_0^2}}{\arccos(x_0/y_0)}. \tag{4}$$

It would be pretty hopeless to solve this problem without invariants. By the way, this is a hard problem by any competition standard.

**E7.** *Each of the numbers* $a_1, \dots, a_n$ *is* 1 *or* $-1$, *and we have*

$$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0.$$

*Prove that* $4 \mid n$.

**Solution.** This is a number theoretic problem, but it can also be solved by invariance. If we replace any $a_i$ by $-a_i$, then $S$ does not change mod 4 since four cyclically adjacent terms change their sign. Indeed, if two of these terms are positive and two negative, nothing changes. If one or three have the same sign, $S$ changes by $\pm 4$. Finally, if all four are of the same sign, then $S$ changes by $\pm 8$.

Initially, we have $S = 0$ which implies $S \equiv 0 \bmod 4$. Now, step-by-step, we change each negative sign into a positive sign. This does not change $S$ mod 4. At the end, we still have $S \equiv 0 \bmod 4$, but also S$= n$, i.e, $4|n$.

**E8.** $2n$ *ambassadors are invited to a banquet. Every ambassador has at most* $n-1$ *enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.*

**Solution.** First, we seat the ambassadors in any way. Let $H$ be the number of neighboring hostile couples. We must find an algorithm which reduces this number whenever $H > 0$. Let $(A, B)$ be a hostile couple with $B$ sitting to the right of $A$ (Fig. 1.3). We must separate them so as to cause as little disturbance as possible. This will be achieved if we reverse some arc $BA'$ getting Fig. 1.4. $H$ will be reduced if $(A, A')$ and $(B, B')$ in Fig. 1.4 are friendly couples. It remains to be shown that such a couple always exists with $B'$ sitting to the right of $A'$. We start in A and go around the table counterclockwise. We will encounter at least $n$ friends of $A$. To their right, there are at least $n$ seats. They cannot all be occupied by enemies of $B$ since $B$ has at most $n-1$ enemies. Thus, there is a friend $A'$ of $A$ with right neighbor $B'$, a friend of $B$.

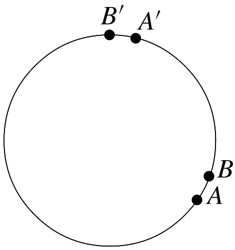

Fig. 1.3. Invert arc $A'B$.

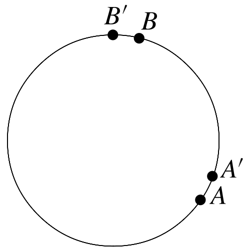

Fig. 1.4

*Remark.* This problem is similar to **E4**, but considerably harder. It is the following theorem in graph theory: *Let G be a linear graph with n vertices. Then G has a Hamiltonian path if the sum of the degrees of any two vertices is equal to or larger than* $n - 1$. In our special case, we have proved that there is even a Hamiltonian circuit.

**E9.** *To each vertex of a pentagon, we assign an integer* $x_i$ *with sum* $s = \sum x_i > 0$. *If* $x, y, z$ *are the numbers assigned to three successive vertices and if* $y < 0$, *then we replace* $(x, y, z)$ *by* $(x + y, -y, y + z)$. *This step is repeated as long as there is a* $y < 0$. *Decide if the algorithm always stops.* (Most difficult problem of IMO 1986.)

**Solution.** The algorithm always stops. The key to the proof is (as in Examples 4 and 8) to find an integer-valued, nonnegative function $f(x_1, \ldots, x_5)$ of the vertex labels whose value decreases when the given operation is performed. All but one of the eleven students who solved the problem found the same function

$$f(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5}(x_i - x_{i+2})^2, \quad x_6 = x_1, \quad x_7 = x_2.$$

Suppose $y = x_4 < 0$. Then $f_{new} - f_{old} = 2sx_4 < 0$, since $s > 0$. If the algorithm does not stop, we can find an infinite decreasing sequence $f_0 > f_1 > f_2 > \cdots$ of nonnegative integers. Such a sequence does not exist.

Bernard Chazelle (Princeton) asked: How many steps are needed until stop? He considered the infinite multiset $S$ of all sums defined by $s(i, j) = x_i + \cdots + x_{j-1}$ with $1 \le i \le 5$ and $j > i$. A multiset is a set which can have equal elements. In this set, all elements but one either remain invariant or are switched with others. Only $s(4, 5) = x_4$ changes to $-x_4$. Thus, exactly one negative element of $S$ changes to positive at each step. There are only finitely many negative elements in $S$, since $s > 0$. The number of steps until stop is equal to the number of negative elements of $S$. We see that the $x_i$ need not be integers.

*Remark.* It is interesting to find a formula with the computer, which, for input $a, b, c, d, e$, gives the number of steps until stop. This can be done without much effort if $s = 1$. For instance, the input $(n, n, 1 - 4n, n, n)$ gives the step number $f(n) = 20n - 10$.

**E10. Shrinking squares. An empirical exploration.** *Start with a sequence* $S = (a, b, c, d)$ *of positive integers and find the derived sequence* $S_1 = T(S) = (|a-b|, |b-c|, |c-d|, |d-a|)$*. Does the sequence* $S$*,* $S_1$*,* $S_2 = T(S_1)$*,* $S_3 = T(S_2), \ldots$ *always end up with* $(0, 0, 0, 0)$*?*

Let us collect material for solution hints:

$$\begin{aligned}&(0, 3, 10, 13) \mapsto (3, 7, 3, 13) \mapsto (4, 4, 10, 10) \mapsto\\ &\quad (0, 6, 0, 6) \mapsto (6, 6, 6, 6) \mapsto (0, 0, 0, 0),\end{aligned}$$

$$\begin{aligned}&(8, 17, 3, 107) \mapsto (9, 14, 104, 99) \mapsto (5, 90, 5, 90) \mapsto\\ &\quad (85, 85, 85, 85) \mapsto (0, 0, 0, 0),\\ &(91, 108, 95, 294) \mapsto (17, 13, 99, 203) \mapsto (4, 86, 104, 186) \mapsto\\ &\quad (82, 18, 82, 182) \mapsto (64, 64, 100, 100) \mapsto (0, 36, 0, 36) \mapsto\\ &\quad (36, 36, 36, 36) \mapsto (0, 0, 0, 0).\end{aligned}$$

Observations:

1. Let $\max S$ be the maximal element of $S$. Then $\max S_{i+1} \le \max S_i$, and $\max S_{i+4} < \max S_i$ as long as $\max S_i > 0$. Verify these observations. This gives a proof of our conjecture.

2. $S$ and $tS$ have the same life expectancy.

3. After four steps at most, all four terms of the sequence become even. Indeed, it is sufficient to calculate modulo 2. Because of cyclic symmetry, we need to test just six sequences $0001 \mapsto 0011 \mapsto 0101 \mapsto 1111 \mapsto 0000$ and $1110 \mapsto 0011$. Thus, we have proved our conjecture. After four steps at most, each term is divisible by 2, after 8 steps at most, by $2^2, \ldots$, after $4k$ steps at most, by $2^k$. As soon as $\max S < 2^k$, all terms must be 0.

In observation 1, we used another strategy, the **Extremal Principle: Pick the maximal element!** Chapter 3 is devoted to this principle.

In observation 3, we used **symmetry.** You should always think of this strategy, although we did not devote a chapter to this idea.

Generalizations:

(a) Start with four real numbers, e.g.,

$$\begin{array}{rrrr}\sqrt{2} & \pi & \sqrt{3} & e\\ \pi-\sqrt{2} & \pi-\sqrt{3} & e-\sqrt{3} & e-\sqrt{2}\\ \sqrt{3}-\sqrt{2} & \pi-e & \sqrt{3}-\sqrt{2} & \pi-e\\ \pi-e-\sqrt{3}+\sqrt{2} & \pi-e-\sqrt{3}+\sqrt{2} & \pi-e-\sqrt{3}+\sqrt{2} & \pi-e-\sqrt{3}+\sqrt{2}\\ 0 & 0 & 0 & 0.\end{array}$$

Some more trials suggest that, even for all nonnegative real quadruples, we always end up with (0, 0, 0, 0). But with $t > 1$ and $S = (1, t, t^2, t^3)$ we have

$$T(S) = [t - 1, (t - 1)t, (t - 1)t^2, (t - 1)(t^2 + t + 1)].$$

If $t^3 = t^2 + t + 1$, i.e., $t = 1.8392867552\ldots$, then the process never stops because of the second observation. This $t$ is unique up to a transformation $f(t) = at + b$.

(b) Start with $S = (a_0, a_1, \ldots, a_{n-1})$, $a_i$ nonnegative integers. For $n = 2$, we reach (0, 0) after 2 steps at most. For $n = 3$, we get, for 011, a pure cycle of length 3: $011 \mapsto 101 \mapsto 110 \mapsto 011$. For $n = 5$ we get $00011 \mapsto 00101 \mapsto 01111 \mapsto 10001 \mapsto 10010 \mapsto 10111 \mapsto 11000 \mapsto 01001 \mapsto 11011 \mapsto 01100 \mapsto 10100 \mapsto 11101 \mapsto 00110 \mapsto 01010 \mapsto 11110 \mapsto 00011$, which has a pure cycle of length 15.

1. Find the periods for $n = 6$ ($n = 7$) starting with 000011 (0000011).

2. Prove that, for $n = 8$, the algorithm stops starting with 00000011.

3. Prove that, for $n = 2^r$, we always reach $(0, 0, \ldots, 0)$, and, for $n \neq 2^r$, we get (up to some exceptions) a cycle containing just two numbers: 0 and evenly often some number $a > 0$. Because of observation 2, we may assume that $a = 1$. Then $| a - b | = a + b \bmod 2$, and we do our calculations in GF(2), i.e., the finite field with two elements 0 and 1.

4. Let $n \neq 2^r$ and $c(n)$ be the cycle length. Prove that $c(2n) = 2c(n)$ (up to some exceptions).

5. Prove that, for odd $n$, $S = (0, 0, \ldots, 1, 1)$ always lies on a cycle.

6. *Algebraization.* To the sequence $(a_0, \ldots, a_{n-1})$, we assign the polynomial $p(x) = a_{n-1} + \cdots + a_0 x^{n-1}$ with coefficients from GF(2), and $x^n = 1$. The polynomial $(1 + x)p(x)$ belongs to $T(S)$. Use this algebraization if you can.

7. The following table was generated by means of a computer. Guess as many properties of $c(n)$ as you can, and prove those you can.

| $n$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c(n)$ | 3 | 15 | 7 | 63 | 341 | 819 | 15 | 255 | 9709 | 63 | 2047 | 25575 |

| $n$ | 27 | 29 | 31 | 33 | 35 | 37 | 39 | 41 | 43 |
|---|---|---|---|---|---|---|---|---|---|
| $c(n)$ | 13797 | 47507 | 31 | 1023 | 4095 | 3233097 | 4095 | 41943 | 5461 |

## Problems

1. Start with the positive integers $1, \ldots, 4n - 1$. In one move you may replace any two integers by their difference. Prove that an even integer will be left after $4n - 2$ steps.

2. Start with the set $\{3, 4, 12\}$. In each step you may choose two of the numbers $a, b$ and replace them by $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you reach the goal (a) or (b) in finitely many steps:

$$\text{(a) } \{4, 6, 12\}, \quad \text{(b) } \{x, y, z\} \text{ with } |x-4|, |y-6|, |z-12| \text{ each less than } 1/\sqrt{3}?$$

3. Assume an $8 \times 8$ chessboard with the usual coloring. You may repaint all squares (a) of a row or column (b) of a $2 \times 2$ square. The goal is to attain just one black square. Can you reach the goal?

4. We start with the state $(a, b)$ where $a$, $b$ are positive integers. To this initial state we apply the following algorithm:

$$\textbf{while } a > 0, \textbf{ do if } a < b \textbf{ then } (a, b) \leftarrow (2a, b-a) \textbf{ else } (a, b) \leftarrow (a-b, 2b).$$

For which starting positions does the algorithm stop? In how many steps does it stop, if it stops? What can you tell about periods and tails?

The same questions, when $a, b$ are positive reals.

5. Around a circle, 5 ones and 4 zeros are arranged in any order. Then between any two equal digits, you write 0 and between different digits 1. Finally, the original digits are wiped out. If this process is repeated indefinitely, you can never get 9 zeros. Generalize!

6. There are $a$ white, $b$ black, and $c$ red chips on a table. In one step, you may choose two chips of different colors and replace them by a chip of the third color. If just one chip will remain at the end, its color will not depend on the evolution of the game. When can this final state be reached?

7. There are $a$ white, $b$ black, and $c$ red chips on a table. In one step, you may choose two chips of different colors and replace each one by a chip of the third color. Find conditions for all chips to become of the same color. Suppose you have initially 13 white 15 black and 17 red chips. Can all chips become of the same color? What states can be reached from these numbers?

8. There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract 1 from each number of a column. Prove that you can reach a table of zeros by a sequence of these permitted moves.

9. Each of the numbers 1 to $10^6$ is repeatedly replaced by its digital sum until we reach $10^6$ one-digit numbers. Will these have more 1's or 2's?

10. The vertices of an n-gon are labeled by real numbers $x_1, \ldots, x_n$. Let $a, b, c, d$ be four successive labels. If $(a-d)(b-c) < 0$, then we may switch $b$ with $c$. Decide if this switching operation can be performed infinitely often.

11. In Fig. 1.5, you may switch the signs of all numbers of a row, column, or a parallel to one of the diagonals. In particular, you may switch the sign of each corner square. Prove that at least one $-1$ will remain in the table.

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 1 | -1 | 1 | 1 |

Fig. 1.5

12. There is a row of 1000 integers. There is a second row below, which is constructed as follows. Under each number $a$ of the first row, there is a positive integer $f(a)$ such that $f(a)$ equals the number of occurrences of $a$ in the first row. In the same way, we get the 3rd row from the 2nd row, and so on. Prove that, finally, one of the rows is identical to the next row.

13. There is an integer in each square of an $8 \times 8$ chessboard. In one move, you may choose any $4 \times 4$ or $3 \times 3$ square and add 1 to each integer of the chosen square. Can you always get a table with each entry divisible by (a) 2, (b) 3?

14. We strike the first digit of the number $7^{1996}$, and then add it to the remaining number. This is repeated until a number with 10 digits remains. Prove that this number has two equal digits.

15. There is a checker at point $(1, 1)$ of the lattice $(x, y)$ with $x, y$ positive integers. It moves as follows. At any move it may double one coordinate, or it may subtract the smaller coordinate from the larger . Which points of the lattice can the checker reach?

16. Each term in a sequence $1, 0, 1, 0, 1, 0, \ldots$ starting with the seventh is the sum of the last 6 terms mod 10. Prove that the sequence $\ldots, 0, 1, 0, 1, 0, 1, \ldots$ never occurs.

17. Starting with any 35 integers, you may select 23 of them and add 1 to each. By repeating this step, one can make all 35 integers equal. Prove this. Now replace 35 and 23 by $m$ and $n$, respectively. What condition must $m$ and $n$ satisfy to make the equalization still possible?

18. The integers $1, \ldots, 2n$ are arranged in any order on $2n$ places numbered $1, \ldots, 2n$. Now we add its place number to each integer. Prove that there are two among the sums which have the same remainder mod $2n$.

19. The $n$ holes of a socket are arranged along a circle at equal (unit) distances and numbered $1, \ldots, n$. For what $n$ can the prongs of a plug fitting the socket be numbered such that at least one prong in each plug-in goes into a hole of the same number (good numbering)?

20. A game for computing $\gcd(a, b)$ and $\mathrm{lcm}(a, b)$.
We start with $x = a$, $y = b$, $u = a$, $v = b$ and move as follows:
if $x < y$ then, set $y \leftarrow y - x$ and $v \leftarrow v + u$
if $x > y$, then set $x \leftarrow x - y$ and $u \leftarrow u + v$
The game ends with $x = y = \gcd(a, b)$ and $(u + v)/2 = \mathrm{lcm}(a, b)$. Show this.

21. Three integers $a, b, c$ are written on a blackboard. Then one of the integers is erased and replaced by the sum of the other two diminished by 1. This operation is repeated many times with the final result 17, 1967, 1983. Could the initial numbers be (a) 2, 2, 2 (b) 3, 3, 3?

22. There is a chip on each dot in Fig. 1.6. In one move, you may simultaneously move any two chips by one place in opposite directions. The goal is to get all chips into one dot. When can this goal be reached?

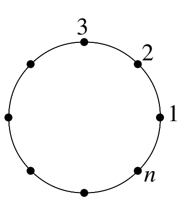


Fig. 1.6

23. Start with $n$ pairwise different integers $x_1, x_2, \ldots, x_n$, $(n > 2)$ and repeat the following step:

$$T : (x_1, \ldots, x_n) \mapsto \left( \frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \ldots, \frac{x_n + x_1}{2} \right).$$

Show that $T, T^2, \ldots$ finally leads to nonintegral components.

24. Start with an $m \times n$ table of integers. In one step, you may change the sign of all numbers in any row or column. Show that you can achieve a nonnegative sum of any row or column. (Construct an integral function which increases at each step, but is bounded above. Then it must become constant at some step, reaching its maximum.)

25. Assume a convex $2m$-gon $A_1, \ldots, A_{2m}$. In its interior we choose a point $P$, which does not lie on any diagonal. Show that $P$ lies inside an even number of triangles with vertices among $A_1, \ldots, A_{2m}$.

26. Three automata $I, H, T$ print pairs of positive integers on tickets. For input $(a, b)$, $I$ and $H$ give $(a + 1, b + 1)$ and $(a/2, b/2)$, respectively. $H$ accepts only even $a, b$. $T$ needs two pairs $(a, b)$ and $(b, c)$ as input and yields output $(a, c)$. Starting with $(5, 19)$ can you reach the ticket (a) $(1, 50)$ (b) $(1, 100)$? Initially, we have $(a, b)$, $a < b$. For what $n$ is $(1, n)$ reachable?

27. Three automata $I, R, S$ print pairs of positive integers on tickets. For entry $(x, y)$, the automata $I, R, S$ give tickets $(x - y, y)$, $(x + y, y)$, $(y, x)$, respectively, as outputs. Initially, we have the ticket $(1, 2)$. With these automata, can I get the tickets (a) $(19, 79)$ (b) $(819, 357)$? Find an invariant. What pairs $(p, q)$ can I get starting with $(a, b)$? Via which pair should I best go?

28. $n$ numbers are written on a blackboard. In one step you may erase any two of the numbers, say $a$ and $b$, and write, instead $(a + b)/4$. Repeating this step $n - 1$ times, there is one number left. Prove that, initially, if there were $n$ ones on the board, at the end, a number, which is not less than $1/n$ will remain.

29. The following operation is performed with a nonconvex non-self-intersecting polygon $P$. Let $A, B$ be two nonneighboring vertices. Suppose $P$ lies on the same side of $AB$. Reflect one part of the polygon connecting $A$ with $B$ at the midpoint $O$ of $AB$. Prove that the polygon becomes convex after finitely many such reflections.

30. Solve the equation $(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$.

31. Let $a_1, a_2, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$. If $n$ is odd, then the product $P = (a_1 - 1)(a_2 - 2) \ldots (a_n - n)$ is even. Prove this.

32. Many handshakes are exchanged at a big international congress. We call a person an *odd person* if he has exchanged an odd number of handshakes. Otherwise he will be called an *even person.* Show that, at any moment, there is an even number of odd persons.

33. Start with two points on a line labeled $0,\ 1$ in that order. In one move you may add or delete two *neighboring* points $(0,\ 0)$ or $(1,\ 1)$. Your goal is to reach a single pair of points labeled $(1,\ 0)$ in that order. Can you reach this goal?

34. Is it possible to transform $f(x) = x^2 + 4x + 3$ into $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$f(x) \mapsto x^2 f(1/x + 1) \quad \text{or} \quad f(x) \mapsto (x - 1)^2 f[1/(x - 1)]?$$