Derek Holton

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A First Step to Mathematical Olympiad Problems

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Vol. 1 A First Step to Mathematical Olympiad Problems *by Derek Holton (University of Otago, New Zealand)*

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To Marilyn, for all her help and encouragement

Foreword

The material in this book was first written for students in New Zealand who were preparing to compete for the six positions in New Zealand's International Mathematical Olympiad (IMO) team. At that stage there was very little mathematical writing available for students who were good at high school mathematics but not yet competent to tackle IMO problems. The aim of the material here then was to give those students sufficient background in areas of mathematics that are commonly the subject of IMO questions so that they were ready for IMO standard work.

This book covers discrete mathematics, number theory and geometry with a final chapter on some IMO problems.

So this book can provide a basis for the initial training of potential IMO students, either with students in a group or for students by themselves. However, I take the approach that solving problems is what mathematics is all about and my second aim is to introduce the reader to what I believe is the essence of mathematics. In many classrooms in many countries, mathematics is presented as a collection of techniques that have to be learnt, often just to be reproduced in examinations. Here I try to present the other, creative, side of the mathematical coin. This is a side that I believe to be far more interesting and exciting. It is also the side that enables students to get some idea of the way that research mathematicians approach their work.

So this book can be used to start students on the trail towards the IMO but its broader aim is to start students on a trail to understanding what mathematics really is and then possibly to taking that understanding and using it in later life, both inside mathematics and outside it.

I would like to thank Irene Goodwin, Leanne Kirk, Lenette Grant, Lee Peng Yee and Zhang Ji for all of their assistance in the preparation of this book.

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Chapter 1
Jugs and Stamps: How To Solve Problems

1.1. Introduction

In this chapter I look at some number problems associated with jugs, consecutive numbers and stamps. I extend and develop these problems in the way that a research mathematician might. At the same time as this is being done, I develop skills of problem solving and introduce some basic mathematical theory, especially about a basic fact of relatively prime numbers.

Whether you are reading this book as a prelude to IMO training or out of interest and curiosity, you should know from the start that mathematics is all about solving problems. Hence the book concentrates on problem solving. Now a problem is only something that, at first sight, you have no idea how to solve. This doesn't mean that a problem is a problem for everyone. Indeed, after you have solved it, it isn't a problem for you any more either. But what I am trying to do here is to both introduce you to some new mathematics and at the same time show you how to tackle a problem that you have no idea at first how to solve.

This book tackles areas of mathematics that are usually not covered in most regular school syllabuses. Sometimes some background is required before getting started but the goal is to show how mathematics is created and how mathematicians solve problems. In the process I hope you, the reader, get a great deal of pleasure out of the work involved in this book.

I have tried to design the material so that it can be worked through by individuals in the privacy of their own brains. But mathematics, like other human pursuits is more fun when engaged in by a group. So let me encourage you to rope in a friend or two to work with you. Friends are also good for talking to about mathematics even if they know nothing about the subject. It's amazing how answers to problems appear when you say your problem out loud.

Now I expect the geniuses amongst you will be able to work through all this book from cover to cover without a break. The mere mortals, however, will most likely read, get stuck somewhere, put the book down (or throw it away) and hopefully go back to it later. Sometimes you'll skim over a difficulty and go back later (maybe much later). But I hope that you will all get some enjoyment out of solving the problems here.

1.2. A Drinking Problem

No problem solving can be done without a problem, so here is the first of many.

Problem 1. *Given a* 3 *litre jug and a* 5 *litre jug can I measure exactly* 7 *litres of water?*

Discussion. You've probably seen this question or one like it before but even if you haven't you can most likely solve it very quickly. Being older and more senile than most of you, bear with me while I slog through it.

I can't see how to get 7. So I'll doodle a while. Hmm. I can make 3, 6 or 9 litres just using the 3 litre jug and 5, 10 or 15 litres with the 5 litre jug. It's obvious, from those calculations that I'm going to have to use both jugs.

Well, it's also pretty clear that $7 \neq 3a + 56$ if I keep a and b positive or zero. So I can't get 7 by just adding water from the two jugs in some combination. So what if I pour water from one jug into another?

Let's fill up the 3 litre jug, then pour the water into the 5 litre jug. I can then fill up the 3 litre jug and pour into the bigger jug again until it's full. That leaves 1 litre in the 3 litre jug. Now if I drink the 5 litres of water from the larger jug I could pour 1 litre of water into some container.

So it's easy. Repeat the performance seven times and we've got a container with 7 litres of water!

Exercises

1. Drink 35 litres of water.
2. Find a more efficient way of producing 7 litres.

   What does it mean by "more efficient"? Does it mean you'll have to drink less or you'll use less water or what?

1.3. About Solving Problems

Now we've seen a problem and worked out a solution, however rough, let's look at the whole business of problem solving. There is no way that at the first reading I can expect you to grasp all the infinite subtleties of the following discussion. So read it a couple of times and move on. But do come back to it from time to time. Hopefully you'll make more sense of it all as time goes on.

Welcome to the Holton analysis of solving problems.

(a) *First take one problem.* Problem solving differs in only one or two respects to mathematical research. The difference is simply that most problems are precisely stated and there is a definite answer (which is known to someone else at the outset). All the steps in between problem and solution are common to both problem solving and research. The extra skill of a research mathematician is learning to pose problems precisely. Of course he/she has more mathematical techniques to hand too.

(b) *Read and understand.* It is often necessary to read a problem through several times. You will probably initially need to read it through two or three times just to get a feel for what's needed. Almost certainly you will need to remind yourself of some details in mid solution. You will definitely need to read it again at the end to make sure you have answered the problem that was actually posed and not something similar that you invented along the way because you could solve the something similar.

(c) *Important words.* What are the key words in a problem? This is often a difficult question to answer, especially on the first reading. However, here is one useful tip. Change a word or a phrase in the problem. If this changes the problem then the word or phrase is important. Usually numbers are important. In the problem of the last section, "jug" is only partially important. Clearly if "jug" was changed to "vase" everywhere, the problem is essentially not changed. However "3" can't be changed to "7" without affecting the problem.

Now you've come this far restate the problem in your own words.

(d) *Panic!* At this stage it's often totally unclear as to what to do next. So, doodle, try some examples, think "have I seen a problem like this before?". Don't be afraid to think "I'll never solve this (expletives deleted) problem". Hopefully you'll get inspiration somewhere. Try another problem. Keep coming back to the one you're stuck on and keep giving it another go. If, after a week, you're still without inspiration, then talk to a friend. Even mothers (who may know nothing about the problem) are marvellous sounding boards. Often the mere act of explaining your difficulties produces an idea or two. However, if you've hit a real toughie, then get in touch with your teacher — that's why they exist. Even then don't ask for a solution. Explain your difficulty and ask for a hint.

(e) *System.* At the doodling stage and later, it's important to bring some system into your work. Tables, charts, graphs, diagrams are all valuable tools. Never throw any of this initial material away. Just as soon as you get rid of it you're bound to want to use it.

Oh, and if you're using a diagram make sure it's a big one. Pokey little diagrams are often worse than no diagram.

And also make sure your diagram covers all possibilities. Sometimes a diagram can lead you to consider only part of a problem.

(f) *Patterns.* Among your doodles, tables and so forth look for patterns. The exploitation of pattern is fundamental to mathematics and is one of its basic powers.

(g) *Guess.* Yes, guess! Don't be afraid to guess at an answer. You'll have to check your guess against the data of the problem or examples you've generated yourself but guesses are the lifeblood of mathematics. OK so mathematicians call their guesses "conjectures". It may sound more sophisticated but it comes down to the same thing in the long run. Mathematical research stumbles from one conjecture (which may or may not be true) to the next.

(h) *Mathematical technique.* As you get deeper into the problem you'll know that you want to use algebraic, trigonometric or whatever techniques. Use what methods you have to. Don't be surprised though, if someone else solves the same problem using some quite different area of mathematics.

(i) *Explanations.* Now you've solved the problem *write out your solution.* This very act often exposes some case you hadn't considered or even a fundamental flaw. When you're happy with your *written* solution, test it out on a friend. Does your solution cover all their objections? If so, try it on your teacher. If not, rewrite it.

My research experience tells me that, at this point, you'll often find a much nicer, shorter, more elegant solution. Somehow the more you work on a problem the more you see through it. It also is a matter of professional pride to find a neat solution.

(j) *Generalisation.* So you may have solved the original problem but now and then you may only have exposed the tip of the iceberg. There may be a much bigger problem lurking around waiting to be solved. Solving big problems is more satisfying than solving little ones. It's also potentially more useful. Have a crack at some generalisations.

In conclusion though, problem solving is like football or chess or almost anything worthwhile. Most of us start off with more or less talent but to be really good you have to practice, practice, practice.

Exercise

3. Look at the steps (a) to (i) and see which of them we went through in the last section with the 3 and 5 litre jugs.

1.4. Rethinking Drinking

How did you go with your 35 litre glug?

Apart from the drinking, there's the question of the unnecessary energy expended.

$$1=2 \times 3 - 1 \times 5.$$

Looking at this equation we can interpret it as "fill the 3 litre jug two times and throw away one lot of 5 litres". "fill" because 2 is positive and "throw away" because —1 is negative.

So

$$7 = 14 \times 3 - 7 \times 5.$$

This means we have to fill the 3 litre jug 14 times and throw away 7 lots of 5 litres! Surely there's a more efficient way? Stop and find one — if you haven't done so already.

OK if you do things the opposite way it's more efficient. Take and fill the 5 litre jug and pour the contents, as far as possible, into the 3 litre jug. Left in the 5 litre jug is a measured 2 litres which you can put into your container. Now fill the 5 litre jug again and add the contents to the container. This gives the 7 litres we wanted and means you only have to drink 3 litres of water.

$$7 = 2 \times 5 - 1 \times 3.$$

With satisfaction you start to move off to another problem. But stop. We've started to see what I was talking about in (i) in the last section. Here we've not just been satisfied with finding a solution. We have been looking for a better solution. Have we found the *best* solution? Think.

Remember $7 = 14 \times 3 - 7 \times 5$.

Notice that $14 = 5 + 9$ and $7 = 3 + 4$. So

$$14 \times 3 - 7 \times 5 = (5 + 9) \times 3 - (3 + 4) \times 5 = 9 \times 3 - 4 \times 5.$$

Filling up the 3 litre jug 9 times is an improvement on our first effort but not as good as our filling up the 5 litre jug twice.

$$
\begin{aligned}
9 \times 3 - 4 \times 5 &= (5 + 4) \times 3 - (3 + 1) \times 5 \\
&= 4 \times 3 - 1 \times 5 \quad \text{(another improvement)} \\
&= (5 - 1) \times 3 - (3 - 2) \times 5 \\
&= 2 \times 5 - 1 \times 3 \quad \text{(our best so far)} \\
&= (3 + 2) \times 5 - (1 + 5) \times 3 \\
&= 5 \times 5 - 6 \times 3 \quad \text{(now it's getting worse)}
\end{aligned}
$$

It's becoming clear that we probably do have the best solution but it will take a little work to prove it.

Let's follow up (j) for a minute. Why stop at 7 litres? Can we produce *m* litres in the container for any positive integer m? That's too easy.

What if we had 3 and 7 litre jugs? Can we put *m* litres of water in our container? What about 3 and 8? What about 3 and *s?* What about *r* and s?

Go on thinking. In the meantime here's a little result in number theory that you should know.

Theorem 1. *Let c and d be positive integers which have no common factors. Then there exist integers a and b such that ac* + *bd* = 1.

In our example with the water we had $c = 3$ and $d = 5$ and we found that $a = 2$ and $b = -1$. But, of course, there are lots of other possible values for *a* and b, so given *c* and d, *a* and *b* are not unique.

Exercises

4. (a) In Theorem 1, let $c = 3$ and $d = 7$. Find possible values for a and b. Can you find *all* possible values for *a* and b?

   (b) Repeat (a) with $c = 4$ and $d = 5$.

5. Given *c* and d, where (c, d) = 1 (c and *d* have no common factor), find all possible *a* and *b* which satisfy the equation $ac + bd = 1$.

6. Given a 3 litre jug and a 5 litre jug what is the best possible way to measure 73 litres into a container? (What do you mean by "best"? Minimum water wasted or minimum number of uses of jugs?)

7. What is the best possible way to get 11 litres of water using only a 3 litre and a 7 litre jug?

8. Show that it is possible to measure any integral number of litres using only a 3 litre and a 7 litre jug.

9. Repeat Exercises 7 and 8 using 4 litre and 13 litre jugs.

10. Is it true that given *r* and s litre jugs, *m* litres of water can be measured for

any positive integer m? (Assume *r* and s are both integers.) Can a best possible solution be found for this problem?

1.5. Summing It Up

Problem 2. *Is it possible to find a sequence of consecutive whole numbers which add up to* 1000? *If so, is the sequence unique?*

Discussion. So we've landed at step (a) again. We've got ourselves another problem.

Working on to (b), what the question asks is can we find numbers a, a+1, a+2 and so on, up to say $a+k$, so that a +(a+1) + (a+2) + - ∠ -+(a+k) equals 1000? When we've done that it wants to know if there's more than one set of consecutive numbers whose sum is 1000.

Moving to step (c) we play "hunt the key words". Well, this question has "consecutive numbers", "add" and "1000". Changing any of these changes the problem. In the follow-up question "unique" is important.

So I understand the problem. Help! I see no obvious way of tackling this at the moment. The solution doesn't appear obvious. Hmm…

Let's see what we can do. Clearly 1000 is too large to handle. Let's get some insight into things by trying for 10 instead.

Well I can do it with *one* consecutive number. Clearly 10 adds up to 10! But I doubt that's what the question is all about. In fact, because it says "numbers" I think it really rules out *one* consecutive number. So we'll work on two or more numbers.

Can we get 10 with two consecutive numbers? Can $a + (a + 1) = 10$? That would mean that $2a + 1 = 10$. Hence $2a = 9$, so $a = 9/2$. But a was supposed to be a whole number, so it can't be a fraction.

Hang on. One of a and $a + 1$ is even while the other is odd. Since the sum of an even and an odd number is odd then we should have known that two consecutive numbers couldn't possibly add up to 10, an even number. (Hmm. Ditto for 1000.)

So what about three numbers? $a +(a +1) + (a + 2) = 10$ gives $3a + 3 = 10$…No solutions folks.

Four numbers? $a +(a +1) + (a + 2) + (a + 3) = 4a + 6 = 10$. Ah, $a = 1$. Yes, 1, 2, 3, 4 do add up to 10.

Five numbers? $a +(a +1) + (a + 2) + (a + 3) + (a + 4) = 10$ gives…Yes, 0, 1, 2, 3, 4 add up to 10.

Six or more numbers clearly won't work. So we see that there are two answers for 10. Will the same thing happen for 1000?

Before you go on you might like to search for "whole numbers" on the web. You'll find that some people accept 0 as a whole number but it doesn't seem to make much sense in this problem. It would be nice not to have both 0+1 + 2 + 3 + 4 and 1 + 2 + 3 + 4. So let's not count 0 among the whole numbers in this book. This also has the virtue of giving a *unique* set of consecutive whole numbers that add up to 10.

Exercise

11. Try Problem 2 with 1000 replaced by (*a*) 20; (b) 30; (c) 40; (d) 100.

Skipping to step (h), I've got the feeling that a little algebra might be useful. We want to find all possible *a* and *k* such that

$$a + (a + 1) + (a + 2) + +(a + k) = 1000. \qquad (1)$$

Trial and error is a possibility. We could try $k = 1$ (two consecutive numbers)…Oh no. We know that two consecutive numbers add up to an odd sum.

Sorry, we could try $k = 2$, then $k = 3$, and so on till we've exhausted all possibilities. But,…I know how to add up the left side of equation (1).

$$a + (a+1) + (a+2) + \cdots + (a+k) = \frac{1}{2}(2a+k)(k+1).^{a}$$

So then has to be solved for *a* and *k*. Has that really made things any easier?

$$(2a + k)(k + 1) = 2000 \qquad (2)$$

Wait a minute. Since $k + 1$ is a factor of the left-hand side of equation (2), it must be a factor of the right-hand side. So $k + 1 = 1, 2, 4, 5, 8, 10,$…Yuk! There seem to be an awful lot of cases.

Of course $k + 1$ is the *number* of consecutive numbers. So we know that $k + 1$ isn't 1 or 2. I suppose that cuts things down a bit.

Exercise

12. Use equation (2) to try Problem 2 with 1000 replaced by
(*a*) 50; (b) 80; (c) 100; (d) 200.
See if there are ways of reducing the number of cases we need to try for k + 1.

Well, I'm not really sure that any of that helped. All we've seen is that some numbers have unique consecutive sets and others have more than one.

But there do seem to be two reasons why we can't solve the *2a* + *k* equation. Either *2a* + *k* is odd and the thing we're equating it to is even or 2*a* + *k* is too big for the right side of its equation. When do those cases occur for our original problem?

Now if *k* + 1 is even, then both *k* and *2a* + *k* are odd. Does 2000 have any *odd* factors? Apart from 1, only 5, 25 and 125. If 2a + *k* = 5, then *k* + 1 = 400. Clearly there's no value for *a* there. If 2a + *k* = 25, then *k* + 1 = 80. Again no solution for a. If 2a + *k* = 125, then *k* + 1 = 16. Ah! Here *a* = 55. This means we get 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70.

But what if *k* + 1 is odd? Then *k* +1 = 5, 2a + 4 = 400 and we get 198, 199, 200, 201, 202 or *k* + 1 = 25, *2a* + 24 = 80 and we get 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52 or *k* + 1 = 125, *2a* + 124 = 16 and we don't get anything.

Ah, that's the key! If *k* + 1 is odd, then *2a* + *k* is even, while if *k* + 1 is even, then 2*a* + *k* is odd. So we have to find the odd factors of 2000 and we also have to find the even factor where the other factor is odd. Once we've worked out that arithmetic then it's all downhill.

Exercises

13. Collect all the solutions to Problem 2.

14. Generalise. (Have a look at odd numbers. Also see if you can find which numbers are the sum of a unique set of consecutive numbers. Are there any numbers that are not the sum of *any* set of consecutive numbers?)

1.6. Licking a Stamp Problem

Problem 3. *The Otohaihai Post Office is in a predicament. It has oodles of* 3 c *and* 5 c *stamps but it has no other stamps at all. What amounts of postage can the Otohaihai post office sell?*

Discussion. Let's look at this problem in the light of the problem solving steps I suggested in Section 1.3.

Well, yes, (*a*) we have a problem. So go to (b) and understand what the question is asking. Isn't this the 3 litre and 5 litre jug problem in disguise?

Follow this idea up in (c). Does it really change the essential nature of Problem 2 if we exchange pence for litres and stamps for jugs? Is there any mathematical difference between stamps and jugs?

Somehow with jugs of water we could "take away". With stamps we can only "add on" or "stick on". If we are trying to get 7 ¢ worth of postage we would need to be able to solve

$$7 = 3a + 5b,$$

where neither *a* nor *b* was ever negative.

There is then, an essential, a mathematically essential, difference between jugs and stamps. We could certainly pour out 7 litres. We certainly *can't* stick down 7 worth.

We may well have reached step (d). If so you may like to go kick a ball, turn on the iPod, watch TV or make yourself a snack. When you've gathered strength move right along to (e).

(Incidentally, this avoidance strategy is well known to mathematicians. We all fervently believe that if we con our brains into thinking they're having a rest, then they mysteriously churn out great new thoughts and theorems. Many of us have woken up in the morning with a pr oblem solved.

It was probably the avoidance strategy of coffee drinking, coupled with the conned-brain syndrome, which prompted Erdös — one of the most prolific mathematicians of the 20th Century[b] — to define a mathematician as someone who turns coffee into theorems.)

OK so back to (e). Rather than using a scattergun approach let's be systematic. It's probably useful to draw up a table at this stage.

Table 1.

amount of postage	1	2	3	4	5	6	7	8	9	10
can be made (√) can't be made (x)	x	x	√	x	√					

Copy and complete the table above. Take the amount of postage up to 25 ¢.

Are there any patterns? We're up to (f) now. Well, of course we can get all multiples of 3 and 5 but we can get a lot of other values too. Obviously 8, 13, 18, etc. can be obtained.

Now to (g). From the data you've compiled what guesses can you make about the amounts of postage you can produce with 3 ¢ and 5 ¢ stamps? If you're arithmetic is correct you should have found that the last cross you have is at 7. From 8 onwards *every* number is ticked. (If you didn't find that, then you'd better go back and see where you went wrong.)

Do you agree with the following guess, or conjecture?

Conjecture 1. *Every amount from* 8 *upwards* can *be obtained.*

Of course, if you agree with the Conjecture, then you must justify your faith. If you don't agree with it, then you have to find a number above 8 that can't be made from 3 and 5.

Exercises

15. If you believe in Conjecture 1, then go on to steps (h) and (i). If you think Conjecture 1 is false, then you have to prove it's false and come up with a conjecture of your own. From there you go on to steps (h) and (i) and possibly back to (g) again.

16. Find an equivalent conjecture to Conjecture 1 with

    (a) 3¢ and 7¢ stamps;
    (b) 3¢ and 11¢ stamps;
    (c) 3¢ and 12¢ stamps.
    Generalise.

17. Repeat Exercise 16 with

    (a) 4¢ and 5¢ stamps;
    (b) 4¢ and 11¢ stamps;
    (c) 4¢ and 6¢ stamps.
    Generalise.

18. Repeat Exercise 16 with

    (a) 6¢ and 7¢ stamps;
    (b) 7¢ and 9¢ stamps;
    (c) 9¢ and 33¢ stamps.
    Generalise.

1.7. A Little Explanation

Conjecture 1 is certainly true. How did you prove it?

This is usually the hardest part of problem solving. The reason is not that it is difficult to write out a proof. Sometimes proofs are easy. No, the reason that proof writing is difficult is that it's not the fun part of problem solving. The fun part is solving the problem. Seeing what the right answer is and "knowing" how you could prove it, is somehow psychologically more interesting than writing out a careful answer.

But I say unto you, he that does not write out a proof has not necessarily solvethed the problem. You only really know you're right when you've safely passed into the haven of step (i).

We've procrastinated long enough. Let's get at it. Well we certainly can do 8, 9,10,11,12,…, 23, 24, 25. Can we do 26? You could work this out from scratch but I've just seen a quicker way. Think a minute. You can get 26 from something you've already produced.

Actually you can get 26 from either 21 or 23 by adding 5 or 3, respectively. And that just about solves the 3 and 5 problem. Because surely 27, 28, 29, 30 and all the rest can be got in exactly the same way from earlier amounts.

So in fact we only have to show that we can get 8 c, 9 c and 10 c After that all the rest follow just by adding enough 3stamps. Conjecture 1 must be true then.

Exercise

19. (*a*) Write out a formal proof of Conjecture 1.

(b) Prove your corresponding conjectures for the amounts in Exercises 16, 17 and 18.

1.8. Tidying Up

Mathematicians like to produce their results with a flourish by calling them *theorems*. These are just statements that can be proved to be true. We'll now present Conjecture 1 as a theorem.

Theorem 2. *All numbers* $n \geq 8$ *can be written in the form* $3a + 5b$*, where* a *and* b *are not negative.*

Proof. First note that 8 = 3 + 5, 9 = 3 x 3 + 0x 5 and 10 = 3 x 0 + 2x 5. If $n \geq 8$, then n is either $8 + 3k$, $9 + 3k$ or $10 + 3k$ for some value of k. Hence if $n > 8$ then either $n = 3(k + 1) + 5$, $3(k + 3)$ or $3k + 2 \times 5$. □

Well at this stage we are still not satisfied. A good mathematician would ask "is 8 best possible?" By that he would mean "is there a number *smaller* than 8 for which Theorem 2 is true?" In other words, is there some $c < 8$ such that for all $n \geq c$ we can express n in the form $3a + 5b$, where a and b are not negative?

But in Table 1 that you completed in Section 1.6, the number 7 should have been given a cross. So clearly there is no number less than 8 which does the job and 8 *is* best possible.

Exercise

20. State and prove the corresponding theorems for your conjectures of Exercise 19(b). In each case show that your results are best possible.

Of course some of you will have realised that we haven't yet completely solved Problem 3, which, after all, asked us to find *all* amounts of postage that can be made with 3 and 5 stamps. We'd better answer that now.

We are going to answer it in the form of a *Corollary.* A corollary is something which follows directly from a result we have just proved. The result below is a simple corollary of Theorem 2 because we can use Theorem 2 plus Table 1 to prove it.

Corollary 1. *If* n = 0,3,5,6 *or any number greater than or equal to* 8, *then* n = 3a + 5b, *where a and b are some non-negative numbers.*

Proof. By Theorem 2, the corollary is true for $n \geq 8$. By Table 1, the corollary is true for $n < 8$. □

It may worry some of you that we included 0 in the list of the Corollary. I have Machiavellian reasons for doing that. These will be revealed in due course.

Exercise

21. State and prove corollaries for all the theorems of Exercise 20.

Table 2.

stamps	c	
3¢,	5¢	8¢
3¢,	7¢	12¢
3¢,	11¢	20¢
3¢,	13¢	...
3¢,	14¢	...
3¢,	16¢	...

1.9. Generalise

We've now built up quite a bit of information about 3and *s*ccombinations (*a*mong other things). For instance we know part of Table 2, where *c* indicates the best possible value in the sense of Theorem 2. In other words, all $n \geq c$ can be obtained and $n = c - 1$ cannot be obtained.

Exercise

22. (*a*) Complete Table 2.

(b) Generalise. In other words, conjecture *c* if you only have 3 and sc stamps.

By now you will have realised that there are essentially two cases for the 3 ¢. and *s*¢. problem. In Exercise 16 you will have come across the problem of whether *s* is divisible by 3 or not. Clearly if *s is* divisible by 3, then you can only ever get amounts which are multiples of 3. Further you can get all multiples of 3. Let's consider what happens when *s* = 12.

Suppose $n = 3a + 12b$, where a and b are not negative. If $a = b = 0$, then $n = 0$. Otherwise $3a + 12b$ is divisible by 3 and so therefore is n. Further if $b = 0$, $n = 3a$. Hence every multiple of 3 can be obtained.

We have thus proved the following lemma.

Lemma 1. *If* $n = 3a + 12b$, *where a and b are not negative, then n must be a multiple of* 3 *and can be* any *multiple of* 3.

The word "lemma" means "little result". When it grows up it could become a theorem. We usually call results lemmas if they are of no intrinsic value but together with other results they do fit together to help prove a theorem.

Usually theorems are results which are important in themselves, like Pythagoras' Theorem, for instance.

Exercise 23. Prove the following lemma.

Lemma 2. *Let s be any multiple of* 3. *If* $n = 3a + sb$, *where a and b are not negative, then n* must *be a multiple of* 3 *and can be* any *multiple of* 3.

But we've strayed from Table 2. Suppose s is not a multiple of 3, what is c? In other words what did you get as your answer to Exercise 22(b)? Can you prove it?

Conjecture 2. $c = 2(s — 1)$.

For $s = 5$ we proved $c = 8$ (Theorem 2) by first showing we could get 8, 9 and 10. After that we just added 3's. The same strategy will work for $s = 7, 11$ and so on (provided s is not a multiple of 3). Can we do the same for s in general? If we can show that we can get 2s — 2, 2s — 1 and 2s using 3's and s's, then we can add on enough 3's and we can get any n.

Well one of this triumvirate of numbers is easy. Surely you don't want me to prove that I can get 2s! So how do you get 2s — 1 and 2s — 2?

Think about s for a minute. When you divide s by 3 you either get a remainder of 1 or a remainder of 2. This means that you can write s either as $3t + 1$ or as $3t + 2$ where $t > 0$. Let's have a look at the case $s = 3t +1$. Now

$$2s — 2 = 6t + 2 — 2 = 6t = 3(2t).$$

Certainly then we can get 2s — 2 in this case because 2s — 2 is just a multiple of 3. So what about 2s — 1? After a bit of thought I'm sure you would have realised that

$$2s — 1 = s + (s — 1) = s + [(3t + 1) — 1] = s + 3t.$$

We can surely get $s + 3t$ using just s's and 3's.